*J. Fluid Mech.* (1976), *vol.* 77, *part* 2, *pp.* 321–354
*Printed in Great Britain*



# Strong MHD helical turbulence and the nonlinear dynamo effect

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(Received 5 November 1975 and in revised form 26 April 1976)

To understand the turbulent generation of large-scale magnetic fields and to advance beyond purely kinematic approaches to the dynamo effect like that introduced by Steenbeck, Krause & Rädler (1966), a new nonlinear theory is developed for three-dimensional, homogeneous, isotropic, incompressible MHD turbulence with helicity, i.e. not statistically invariant under plane reflexions. For this, techniques introduced for ordinary turbulence in recent years by Kraichnan (1971*a*), Orszag (1970, 1976) and others are generalized to MHD; in particular we make use of the eddy-damped quasi-normal Markovian approximation. The resulting closed equations for the evolution of the kinetic and magnetic energy and helicity spectra are studied both theoretically and numerically in situations with high Reynolds number and unit magnetic Prandtl number.

Interactions between widely separated scales are much more important than for non-magnetic turbulence. Large-scale magnetic energy brings to equipartition small-scale kinetic and magnetic excitation (energy or helicity) by the 'Alfvén effect'; the small-scale 'residual' helicity, which is the difference between a purely kinetic and a purely magnetic helical term, induces growth of large-scale magnetic energy and helicity by the 'helicity effect'. In the absence of helicity an inertial range occurs with a cascade of energy to small scales; to lowest order it is a $-\frac{3}{2}$ power law with equipartition of kinetic and magnetic energy spectra as in Kraichnan (1965) but there are $-2$ corrections (and possibly higher ones) leading to a slight excess of magnetic energy. When kinetic energy is continuously injected, an initial seed of magnetic field will grow to approximate equipartition, at least in the small scales. If in addition kinetic helicity is injected, an inverse cascade of magnetic helicity is obtained leading to the appearance of magnetic energy and helicity in ever-increasing scales (in fact, limited by the size of the system). This inverse cascade, predicted by Frisch *et al.* (1975), results from a competition between the helicity and Alfvén effects and yields an inertial range with approximately $-1$ and $-2$ power laws for magnetic energy and helicity. When kinetic helicity is injected at the scale $l_{\text{inj}}$ and the rate $\tilde{\epsilon}^V$ (per unit mass), the time of build-up of magnetic energy with scale $L \gg l_{\text{inj}}$ is $t \approx L(|\tilde{\epsilon}^V| l_{\text{inj}}^2)^{-\frac{1}{3}}$.

## 1. Introduction

*Motivations*

Turbulence and magnetic fields ($M$-fields) are a common feature of many celestial bodies. $M$-fields are very often observed on the largest available scales whereas turbulence, at least the most energetic part of it, is more frequently confined to smaller scales. In the last decade, several attempts have been made to explain large-scale $M$-field generation (or regeneration) as a consequence of the probable lack of reflexional symmetry of small-scale turbulence in situations involving rotation and stratification (Parker 1955; Steenbeck *et al.* 1966; Moffatt 1970*a*, *b*; see also review paper of Moffatt 1973). The lack of reflexional symmetry is measured by the kinetic helicity $\frac{1}{2}\langle \mathbf{v}.(\nabla\wedge\mathbf{v})\rangle$ of the small-scale turbulent velocity field ($V$-field). The destabilization of large-scale $M$-fields by kinetic helicity ($V$-helicity), the 'helicity or $\alpha$ effect', is now fairly well understood on the basis of the linear equation of advection of the $M$-field:†

$$\left.\begin{aligned}(\partial/\partial t-\lambda\nabla^2)\,\mathbf{b} &= -(\mathbf{v}.\nabla)\,\mathbf{b}+(\mathbf{b}.\nabla)\,\mathbf{v},\\ \nabla.\mathbf{b} &= 0.\end{aligned}\right\} \tag{1.1}$$

The growth of the large-scale $M$-field raises the question of its back-reaction on the $V$-field through the Lorentz force term of the momentum equation for an incompressible fluid:

$$\left.\begin{aligned}(\partial/\partial t-\nu\nabla^2)\,\mathbf{v} &= -(\mathbf{v}.\nabla)\,\mathbf{v}+(\mathbf{b}.\nabla)\,\mathbf{b}-\nabla p+\mathbf{f},\\ \nabla.\mathbf{v} &= 0.\end{aligned}\right\} \tag{1.2}$$

In a strongly turbulent medium the search for a saturation mechanism which will prevent indefinite growth requires a theory of helical magnetohydrodynamic (MHD) turbulence which takes into account all nonlinear interactions and their conserved quantities such as total (kinetic plus magnetic) energy, magnetic helicity ($M$-helicity) and cross-helicity. In Frisch *et al.* (1975), henceforth referred to as I, some of the consequences of the conservation of $M$-helicity $\frac{1}{2}\langle \mathbf{a}.\mathbf{b}\rangle$ ($\mathbf{a}$ = vector potential of $M$-field $\mathbf{b}$) were worked out; the analysis of helical MHD absolute equilibrium ensembles has led to the conjecture that an 'inverse' cascade of $M$-helicity from small to large scales exists when helicity is injected at small scales. Absolute equilibrium, which, by definition, has all dissipation removed and allows only a finite wavenumber band, may be useful in predicting the direction of cascades (Léorat 1975, § 2.7) but is certainly very far from true turbulence, which, even in the stationary case, is essentially a non-equilibrium situation.

*Tools*

A reasonably accurate quantitative theory of non-magnetic turbulence is a prerequisite for a serious theory of MHD turbulence. From this standpoint, we

† All the notation is taken from Frisch *et al.* (1975). Notice that the $V$- and $M$-energy and helicity spectra previously denoted by $E_K(k)$, $H_K(k)$, $E_M(k)$ and $H_M(k)$, are now denoted by $E_k^V$, $H_k^V$, $E_k^M$ and $H_k^M$. The corresponding injection spectra are denoted by $F_k^V$, $\tilde{F}_k^V$, $F_k^M$ and $\tilde{F}_k^M$ and their integrals, the rates of injection, by $\epsilon^V$, $\tilde{\epsilon}^M$, $\epsilon^V$ and $\tilde{\epsilon}^M$.

believe that the closely related test-field model (TFM) of Kraichnan (1971*a*; cf. also Sulem, Lesieur & Frisch 1975) and the eddy-damped quasi-normal Markovian (EDQNM) approximation of Orszag (1970, 1976) represent a breakthrough for homogeneous isotropic turbulence. It may be useful to recall some of the results obtained so far. Both theories give a $-\frac{5}{3}$ inertial range for three-dimensional turbulence and the TFM gives a Kolmogorov constant in excellent agreement with measurements (Kraichnan 1971*b*). In the two-dimensional case, a $-3$ inertial range is obtained which is also well supported experimentally (Kraichnan 1971*b*; Pouquet *et al.* 1975). Their application to the two- and three-dimensional error-growth problem is important not only for weather prediction but also for the understanding of the stochastic nature of turbulence itself (Lorenz 1969; Leith 1971; Leith & Kraichnan 1972). Moreover, the theories have been tested against direct numerical simulation of the Navier–Stokes equations; no serious discrepancies in single-time moments have been found at the highest Reynolds number which can be reached on present computers (Orszag & Patterson 1972; Herring *et al.* 1974). As a noteworthy practical feature, these theories lead to a set of integro-differential equations for the spectra which are easily integrated numerically even at huge Reynolds numbers (Pouquet *et al.* 1975; André & Lesieur 1975).

*Scope*

The general framework of the present investigation is now defined. Homogeneous, isotropic, helical MHD turbulence is assumed, i.e. we consider random solutions of (1.1) and (1.2) which are statistically invariant under translations and rotations, but not under plane reflexions. The $M$-field is taken to be statistically invariant under sign reversal ($\mathbf{b} \rightarrow -\mathbf{b}$) at the initial time; the equations of motion then imply that it remains so and that the cross-helicity ($V$- and $M$-field correlations) is permanently zero. Since our main interest is in the generation of $M$-fields from vanishingly small seed fields, this is no serious restriction. The viscosity $\nu$ and the magnetic diffusivity $\lambda$ can take arbitrary values in the general formalism but most of the theory is then worked out for cases where the Reynolds number is large and the magnetic Prandtl number $\nu/\lambda$ is unity. The investigation of the small scales of MHD turbulence, where helicity is not particularly important but where the influence of the magnetic Prandtl number is strongly felt, is beyond the scope of this paper (cf. also Kraichnan & Nagarajan 1967).

The most serious restriction from the standpoint of the physicist who tries to understand stellar or planetary $M$-fields is the assumption of homogeneity and isotropy. However, since the physics of strong helical MHD turbulence are almost completely unexplored, it may be wise first to analyse in detail the simplest case. Using a method as free as presently possible from *ad hoc* phenomenological assumptions, we have tried to understand the basic mechanism for the growth of large-scale $M$-fields; this aspect may be more important than the attempt made at the end of § 6 to evaluate the time of regeneration of the solar magnetic field.

In the last section, the reader will find a summary of the paper. If interested only in the applications of the spectral equations he may skip their derivation in § 2. In § 3, the effects of non-local interactions are studied in detail; § 4, on the $-\frac{3}{2}$

inertial range, may be read independently; §§5 and 6 deal with the inverse cascade of $M$-helicity and the nonlinear turbulent dynamo. For the understanding of the general formalism a reasonable familiarity with the theory of homogeneous turbulence as outlined, for example, in Orszag (1976) is helpful. Knowledge of at least the first section of I is also assumed to avoid repetition of definitions. Finally, the reader may find a version with more introductory material in Léorat (1975).

## 2. The spectral equations of helical MHD turbulence

### *Eddy-damped quasi-normal Markovian approximation*

Our aim is to construct a modified eddy-damped quasi-normal Markovian (EDQNM) approximation for MHD helical turbulence. For a detailed exposition of the usual EDQNM theory, the reader is referred to Orszag (1970, 1976; cf. also Sulem *et al.* 1975). The essential steps may be outlined as follows. Let the Navier–Stokes or the MHD equations be written symbolically as

$$du/dt = uu,$$

where $u$ stands for the unknown functions ($\mathbf{v}$ or $\mathbf{v}$ and $\mathbf{b}$) and where $uu$ represents all the nonlinear terms. This very contracted notation is used only to bring out quadratic nonlinearity. The linear (dissipative) and forcing terms have been dropped since they pose no particular closure problem and can easily be reintroduced at the end. Assuming that the first moment $\langle u\rangle$ is zero, we obtain for the second and third moments (still in symbolic form)

$$d\langle uu\rangle/dt = \langle uuu\rangle, \quad d\langle uuu\rangle/dt = \langle uuuu\rangle. \qquad (2.1), (2.2)$$

The quasi-normal approximation replaces $\langle uuuu\rangle$ by its Gaussian value, which is a sum of products of second-order moments. This approximation suffers from well-known defects which can be cured by the introduction of a suitable linear relaxation operator $\mu$ of triple correlations (a procedure called *eddy damping*) on the right-hand side of (2.2) and by 'Markovianization'. One finally obtains a closed equation for simultaneous second-order moments

$$d\langle u(t)\,u(t)\rangle/dt = \theta(t)\,\langle u(t)\,u(t)\rangle\,\langle u(t)\,u(t)\rangle, \qquad (2.3)$$

where the triad-relaxation time $\theta(t)$ (here an operator) is defined as

$$\theta(t) = \int_0^t d\tau \exp\left\{-\int_\tau^t \mu(s)\,ds\right\}. \qquad (2.4)$$

Notice that for short times $\theta(t) = t + O(t^2)$ and for the stationary case

$$(\mu = \text{constant}) \quad \theta = \mu^{-1}.$$

The eddy-damping operator $\mu$ may be obtained either from a phenomenological study or from the analysis of an auxiliary problem as in the TFM of Kraichnan (1971*a*). An important task will be to find the appropriate eddy-damping operator for MHD turbulence.

The realizability (e.g. the positivity of energy spectra) of the EDQNM equations (2.3) is ensured by the existence of a stochastic model (Kraichnan 1971*a*; Frisch, Lesieur & Brissaud 1974).

The general EDQNM equations can now be given explicitly in operator form. Let the basic Navier–Stokes or MHD equations be written as

$$du(t)/dt = L(u(t), u(t)) + L_0 u(t) + f(t); \tag{2.5}$$

$L(.,.)$ collects all the quadratic terms, $L_0$ stands for the linear dissipative terms and $f(t)$, the forcing, is a white noise in time with second-order moments given by

$$\langle f(t) \otimes f(t') \rangle = F\delta(t-t'), \tag{2.6}$$

where $F$ is a prescribed forcing tensor. With these assumptions the EDQNM approximation reads

$$\begin{aligned} d\langle u \otimes u \rangle/dt = 4\theta\{&L(u,u) \otimes L(u,u) + L(L(u,u),u) \otimes u \\ &+ u \otimes L(u, L(u,u))\} + \langle L_0 u \otimes u \rangle + \langle u \otimes L_0 u \rangle + F, \end{aligned} \tag{2.7}$$

where $u$-factors belonging to the same moments have been linked together. When the triad-relaxation time $\theta$ is just a constant, one recovers the Markovian random coupling (MRC) equations of Frisch *et al.* (1974).

*Eddy-damping rates for MHD*

We turn now to the problem of the turbulent MHD equations. As explained in the introduction we shall assume homogeneity, isotropy and no cross-correlations between the $V$-field and $M$-field (in particular no cross-helicity). The EDQNM approximation will lead to a set of four integro-differential equations for the $V$- and $M$-energy and helicity spectra, denoted by $E_k^V$, $E_k^M$, $H_k^V$ and $H_k^M$. Their derivation requires (i) the determination of the triad-relaxation time $\theta$, involving the eddy-damping operator $\mu$, and (ii) the determination of the explicit form of the right-hand side of (2.7) with homogeneity and isotropy taken into account. The second step is well defined but requires very extensive algebra although it is basically the same as in the case of non-magnetic non-helical turbulence, which is described in detail in Leslie (1973).

As for non-magnetic homogeneous turbulence, the eddy-damping operator is diagonal in the Fourier representation and must be calculated for each triad of wavenumbers $(k, p, q)$; one takes as usual

$$\mu_{kpq} = \mu_k + \mu_p + \mu_q, \tag{2.8}$$

where the $\mu_k$'s are called *eddy-damping rates*. The form (2.8) implies complete symmetry of $\mu_{kpq}$, and hence of $\theta_{kpq}$, the triad-relaxation time, with respect to $k, p, q$; this in turn ensures the conservation of all quadratic invariants. We have chosen the following expression for the eddy-damping rate:

$$\begin{aligned} \mu_k &= C_S \left[ \int_0^k q^2 (E_q^V + E_q^M)\, dq \right]^{\frac{1}{2}} + C_A k \left[ 2 \int_0^k E_q^M\, dq \right]^{\frac{1}{2}} + (\nu + \lambda) k^2 \\ &= \mu_k^S + \mu_k^A + \mu_k^D. \end{aligned} \tag{2.9}$$

The first term $\mu_k^S$ corresponds to the self-distortion or nonlinear scrambling of the flow, the second term $\mu_k^A$ to the effect of Alfvén waves and the third $\mu_k^D$ to viscous and Joule dissipation.

In the absence of an $M$-field the self-distortion term $\mu_k^S$ represents the rate at which scales of wavenumber $k$ are being distorted by larger scales. The simplified form $\mu_k^S \sim (k^3 E_k^V)^{\frac{1}{2}}$ used by Orszag (1970) is inappropriate for initial-value problems, as noticed by Pouquet *et al.* (1975). The numerical constant $C_S$ can be adjusted to yield agreement with the Kolmogorov constant $C_{\text{Kol}}$ determined either experimentally or from the TFM. For $C_{\text{Kol}} = 1{\cdot}4$ the correct value is $C_S = 0{\cdot}36$; this value also gives a skewness factor of 0·39 in excellent agreement with experimental results (see, for example, Batchelor 1953, figure 6.3) for Reynolds numbers based on the Taylor microscale of the order of thirty (André & Lesieur 1976). Since we assumed uncorrelated $V$- and $M$-fields (no cross-helicity) we have, in the MHD case, simply added a magnetic contribution to $\mu_k^S$.

The essential new term is the Alfvén eddy-damping rate $\mu_k^A$. It is known that a large-scale random $M$-field of r.m.s. value $b_0$ relaxes triple correlations† with wavenumbers $\sim k$ in a time $\sim (kb_0)^{-1}$, which is the time for an Alfvén wave of speed $b_0$ to travel a distance $\sim k^{-1}$ (Kraichnan 1965; Orszag & Kruskal 1968). If one linearizes the MHD equations around the large-scale random $M$-field, the decorrelation may be understood as due to $\mathbf{v}+\mathbf{b}$ and $\mathbf{v}-\mathbf{b}$ travelling as Alfvén waves in opposite directions. For a Gaussian large-scale $M$-field an explicit calculation then yields $C_A = 1/\sqrt{3}$. We have used

$$\left[2\int_0^k E_q^M \, dq\right]^{\frac{1}{2}}$$

instead of $b_0$ because only scales larger than $k^{-1}$ contribute to the Alfvén relaxation. It is easily checked that omission of the $\mu_k^A$ term would deeply effect the dynamics of MHD turbulence; indeed, in an inertial range with $E_k^V \approx E_k^M \sim k^{-n}$ the self-distortion rate behaves like $k^{\frac{1}{2}(3-n)}$ and the Alfvén rate like $b_0 k$; if $n > 1$, the Alfvén rate dominates at large wavenumbers.

In the non-magnetic case the dissipative contribution $\mu_k^D$ is $\nu k^2$. Our choice of $(\nu+\lambda)k^2$ may appear questionable since it seems more appropriate to take only $\nu k^2$ if $\mu_k$ is used in the relaxation of a purely kinetic triple correlation as is done by Nagarajan (1971). However, $\mu_k^D$ will become important only in the dissipation range and we know that small-scale $V$- and $M$-excitations are strongly coupled by Alfvén waves.

*Remark* (2.1). The reader familiar with the test-field model (TFM) of Kraichnan (1971*a*; see also Sulem *et al.* 1975) may believe that it gives, at least in principle, a procedure by which all eddy-damping rates may be unambiguously determined. A distinctive feature of the TFM in the non-magnetic case is the removal of spurious interactions with large-scale $V$-fields in order to ensure Galilean invariance. In the magnetic case, interactions with large-scale $M$-fields should not be removed if Alfvén eddy damping is to be kept: otherwise a $-\frac{5}{3}$ instead of a $-\frac{3}{2}$ inertial range will be obtained (see § 4). With this in mind, a magnetic

† More precisely, those triple correlations responsible for nonlinear transfer.

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right) E_k^V = F_k^V + \int_{\Delta_k} dp\,dq\,\theta_{kpq}(T_{VV}^V + T_{\tilde{V}\tilde{V}}^V + T_{VM}^V + T_{MM}^V + T_{\tilde{M}\tilde{M}}^V)$$

$$\left(\frac{\partial}{\partial t}+2\lambda k^2\right) E_k^M = F_k^M + \int_{\Delta_k} dp\,dq\,\theta_{kpq}(T_{VM}^M + T_{\tilde{V}\tilde{M}}^M + T_{MM}^M + T_{\tilde{M}\tilde{M}}^M)$$

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right) H_k^V = \tilde{F}_k^V + \int_{\Delta_k} dp\,dq\,\theta_{kpq}(T_{V\tilde{V}}^{\tilde{V}} + T_{\tilde{V}M}^{\tilde{V}} + T_{M\tilde{M}}^{\tilde{V}})$$

$$\left(\frac{\partial}{\partial t}+2\lambda k^2\right) H_k^M = \tilde{F}_k^M + \int_{\Delta_k} dp\,dq\,\theta_{kpq}(T_{V\tilde{M}}^{\tilde{M}} + T_{\tilde{V}M}^{\tilde{M}} + T_{M\tilde{M}}^{\tilde{M}})$$

$$T_{VV}^V = kp^{-1}q^{-1}b_{kpq}(k^2E_p^VE_q^V - p^2E_q^VE_k^V) \quad T_{\tilde{V}\tilde{V}}^V = -p^{-1}q^{-2}c_{kpq}(k^2H_p^VH_q^V - p^2H_q^VH_k^V)$$

$$T_{VM}^V = kpq^{-1}c_{kpq}E_q^ME_k^V$$

$$T_{MM}^V = k^3p^{-1}q^{-1}c_{kpq}E_p^ME_q^M, \quad T_{\tilde{M}\tilde{M}}^V = -(\tfrac{1}{2})k^3p^{-1}qh_{kpq}H_p^MH_q^M$$

$$T_{VM}^M = k^5p^{-3}q^{-1}c_{kpq}E_p^VE_q^M + kp^{-1}q^{-1}h_{kpq}(k^2E_p^ME_q^V - p^2E_q^VE_k^M)$$

$$T_{\tilde{V}\tilde{M}}^M = kp^{-1}qh_{kpq}(k^2p^{-2}H_p^VH_q^M - p^2q^{-2}H_q^VH_k^M)$$

$$T_{MM}^M = -k^3p^{-1}q^{-1}c_{kpq}E_q^ME_k^M, \quad T_{\tilde{M}\tilde{M}}^M = kp^2e_{kpq}H_q^MH_k^M$$

$$T_{V\tilde{V}}^{\tilde{V}} = kp^{-1}q^{-1}b_{kpq}(k^2H_p^VE_q^V - p^2E_q^VH_k^V) - k^2pq^{-2}c_{kpq}(E_p^VH_q^V - H_q^VE_k^V)$$

$$T_{\tilde{V}M}^{\tilde{V}} = -kpq^{-1}c_{kpq}E_q^MH_k^V, \quad T_{M\tilde{M}}^{\tilde{V}} = k^4q^{-1}f_{kpq}H_p^ME_q^M$$

$$T_{V\tilde{M}}^{\tilde{M}} = kp^{-1}q^{-1}h_{kpq}(k^2H_p^ME_q^V - p^2E_q^VH_k^M)$$

$$T_{\tilde{V}M}^{\tilde{M}} = kp^{-1}q^{-1}h_{kpq}(k^2p^{-2}H_p^VE_q^M - p^2k^{-2}H_q^VE_k^M)$$

$$T_{M\tilde{M}}^{\tilde{M}} = p^2k^{-1}e_{kpq}H_q^ME_k^M - kpq^{-1}j_{kpq}E_q^MH_k^M$$

TABLE 1. Spectral equations of helical MHD turbulence. The notation is defined in table 2.

$\Delta_k$ is a subset of the $p$, $q$ plane such that $k$, $p$ and $q$ can form a triangle

$$b_{kpq} = pk^{-1}(xy+z^3), \quad j_{kpq} = pk^{-1}z(1-x^2)$$

$$c_{kpq} = pk^{-1}z(1-y^2), \quad e_{kpq} = x(1-z^2)$$

$$h_{kpq} = 1-y^2, \quad f_{kpq} = z-xy-2zy^2$$

$$\theta_{kpq}(t) = \{1-\exp(-\mu_{kpq}t)\}/\mu_{kpq}, \quad \mu_{kpq} = \mu_k+\mu_p+\mu_q$$

$$\mu_k = (\nu+\lambda)k^2 + C_s\left(\int_0^k q^2(E_q^V+E_q^M)\,dq\right)^{\frac{1}{2}} + (1/\sqrt{3})\,k\left(2\int_0^k E_q^M\,dq\right)^{\frac{1}{2}}$$

$C_s = 0{\cdot}36$ gives Kolmogorov constant of 1·4 in absence of magnetic field

TABLE 2. Eddy-relaxation time and geometric coefficients appearing in the spectral equations.

version of the TFM can probably be constructed but would lead to a rather dramatic increase in algebraic complexity, particularly in the helical case. So mainly for the sake of simplicity, we decided to use the simpler, phenomenologically based, EDQNM. It may be interesting to point out that the new *qualitative* results obtained in this paper, like the existence of the magnetic helicity effect and of the inverse cascade of magnetic helicity, are practically insensitive to the precise form of the eddy-damping rate as long as $\theta_{kpq}$ remains positive and completely symmetrical to ensure realizability and energy and helicity conservation.

*Explicit form of the spectral equations*

In table 1 the reader will find the spectral equations of helical MHD turbulence. The notation is defined in table 2. The functions $F_k^V$, $\tilde{F}_k^V$, $F_k^M$ and $\tilde{F}_k^M$ are prescribed injection spectra of $V$-energy, $V$-helicity, $M$-energy and $M$-helicity which must satisfy the realizability conditions (cf. equations (14) and (15) of I)

$$|\tilde{F}_k^V| \leqslant kF_k^V, \quad |\tilde{F}_k^M| \leqslant F_k^M/k. \tag{2.10}$$

$\Delta_k$ denotes the domain of the $p$, $q$ plane such that $k$, $p$ and $q$ can be the lengths of the sides of a triangle, the cosines of the interior angles being $x$, $y$ and $z$. The simplified form of the triad-relaxation time $\theta_{kpq}(t)$, taken from Leith (1971) and Leith & Kraichnan (1972), is chosen to agree with (2.4) both for short times and in the stationary case without requiring integration over past spectra.

In the helical non-magnetic case, the spectral equations are the same as in André & Lesieur (1976). In the absence of helicity the equations reduce, except in the form of the triad-relaxation time, to the MHD spectral equations of Nagarajan (1971), which in turn are just a Markovian version of Kraichnan's (1958) direct-interaction-approximation (DIA) equations. In particular, the geometrical coefficients $b_{kpq}$, $c_{kpq}$, $h_{kpq}$ and $j_{kpq}$ are the same as in Kraichnan & Nagarajan (1967). In Nagarajan (1971) a different choice of the triad-relaxation time $\theta_{kpq}$ is made; this choice is not symmetrical in $k$, $p$ and $q$, so that energy conservation does not hold; furthermore the non-local contribution to the eddy-damping rate is there taken to be $v_0 k$ (where $v_0$ is a r.m.s. velocity) instead of $b_0 k$; this leads to spurious effects of large-scale $V$-energy on small-scale $M$-excitation.

*Conservation properties*

The spectral equations have the same conservation properties as the original MHD equations: conservation of total energy and of $M$-helicity. These conservation laws can be formulated in two different ways. One formulation puts the viscosity $\nu$ and magnetic diffusivity $\lambda$ equal to zero and then writes, at least formally (see following remark),

$$\frac{d}{dt}\int_0^\infty (E_k^V + E_k^M)\,dk = 0, \quad \frac{d}{dt}\int_0^\infty H_k^M\,dk = 0. \tag{2.11}$$

The derivation requires interchanging integration variables and makes use of the complete symmetry of $\theta_{kpq}$ and some trigonometric identities which can be found in the appendix. The other formulation of conservation is to keep $\nu$ and $\lambda$ and to show that the transfer of total energy and of magnetic helicity integrates to zero (transfer is defined as the contribution of the nonlinear terms to the right-hand side of the spectral equations).

*Remark* (2.2). For the interchange of certain integrals required in the proof of conservation, sufficiently fast decrease at large wavenumbers of the spectra is required. This smoothness property of the solution of the non-dissipative spectral equations, if it holds initially, can be shown to persist for a finite time as in the non-magnetic non-helical case (Lesieur 1973; Lesieur & Sulem 1976). After this time, which is of the order of the inverse of the r.m.s. initial vorticity, singularities may appear in the spectral equations and presumably also in the true MHD equations. For positive $\nu$ and $\lambda$ smoothness persists forever (Bardos *et al.* 1976).

Another important property of the spectral equations is the existence of *absolute-equilibrium* solutions which are the same as for the original MHD equations (cf. I, equations (30)–(32) with $\phi = 0$). For this, $\nu$, $\lambda$ and injection spectra must be taken as zero and the spectra restricted to a finite spectral band ($k_{\min}$, $k_{\max}$). Checking the spectral equations for conservation and for absolute-equilibrium solutions constitutes a particularly severe test of the accuracy of the algebra.

*Numerical integration*

The spectral equations given in table 1 are well suited for numerical integration with standard techniques described, for example, in Leith & Kraichnan (1972). Reynolds numbers and, more generally, ratios of maximum to minimum wavenumber much higher than for direct numerical simulations (Pouquet & Patterson 1976) can be reached. Particular attention must, however, be paid to the following point: the numerical technique is based on a logarithmic subdivision of the $k$ axis of the form $k_L \sim 2^{L/F}$; as noticed by Leith & Kraichnan (1972) and Pouquet *et al.* (1975), this has the consequence that all 'non-local' interactions will be cut off by the method of integration when they involve non-isosceles triads of wavenumbers ($k$, $p$, $q$) such that the ratio of smallest to middle wavenumber is less than $a = 2^{1/F} - 1$. This is particularly unacceptable in MHD turbulence because, as we shall see in the following sections, the physically most interesting interactions responsible for the Alfvén and helicity effects are highly non-local. This difficulty can be overcome by using, for example, the refined mesh technique of Leith & Kraichnan (1972). We choose an alternative method, first introduced by Pouquet *et al.* (1975): direct numerical calculation of the transfer terms by the standard technique is done only for local interactions with wavenumber ratios larger than $a$; the non-local contributions to transfer are expanded in powers of the wavenumber ratio and truncated in such a way as to satisfy the various conservation relations while keeping the most interesting physical effects. The resulting rather simple expressions which will be found in the next section are then calculated numerically together with the local terms.

## 3. Non-local interactions: Alfvén and helicity effects

The complete spectral equations of MHD helical turbulence given in table 1 have a rather complicated integro-differential structure which prevents easy theoretical investigation. In this section, we shall concentrate on *non-local* effects corresponding to situations where, in the interacting triad ($k$, $p$, $q$), one of the following two conditions is fulfilled:

(i) $q \ll k \sim p$ or $p \ll k \sim q$,

(ii) $k \ll p \sim q$.

If we associate the wavenumber $k$ with typical scales $\sim k^{-1}$, then (i) and (ii) correspond to the case where very large or very small scales are acting on the scale $k^{-1}$. Non-local effects are known to be important in two-dimensional non-magnetic turbulence (Kraichnan 1971*b*; Pouquet *et al.* 1975). The importance for magnetic non-helical three-dimensional turbulence of type (i) effects was recognized by Kraichnan (1965), who noticed that, in the presence of large-scale $M$-energy, Alfvén waves can bring small-scale $V$- and $M$-energy to equipartition and relax triple correlations in a time which may be shorter than the local eddy-turnover time.† In helical magnetic turbulence, type (ii) non-local effects are to be expected in view of the results of Steenbeck *et al.* (1966) and Moffatt (1970*a*, *b*) concerning the destabilization of large-scale $M$-fields by small-scale helicity; this is the helicity effect, also called the $\alpha$-effect.

We shall denote by $(\partial E_k^V/\partial t)_{\text{Loc}}$ the contributions to the derivative of the $V$-energy spectrum arising from dissipation, forcing, and those nonlinear interactions involving triads $(k, p, q)$ for which the ratio of smallest to middle wavenumber is less than $a$, where $a$ is a small expansion parameter; similarly, we have $(\partial E_k^M/\partial t)_{\text{Loc}}$, etc. The non-local contributions are denoted by $(\partial E_k^V/\partial t)_{\text{NLoc}}$ and separated into $(\partial E_k^V/\partial t)_{NS}$ and $(\partial E_k^V/\partial t)_{SS}$, the respective contributions of type (i) and type (ii); $LS$ and $SS$ stand for large-scale and small-scale. We then systematically expand the transfer terms in powers of $a$ and look for contributions of order zero and one (only first-order eddy-diffusivity contributions remain in the non-magnetic case). If, for example, $q \ll p \sim k$ we write the integrals in polar co-ordinates using as variables $q$ and the angle $(\mathbf{q}, \mathbf{k})$ and use the expansions of $p$, the volume element and the geometric coefficients given in the appendix; the eddy-damping rate $\mu_p$ is expanded in Taylor series near $k$ but, eventually, this gives no contribution to order one; notice that $\mu_q$ is not expanded but kept as it stands because its order in $a$ is not known.

After expansion, we obtain

$$
\begin{array}{ll|l}
(\partial E_k^V/\partial t)_{\text{NLoc}} & = -k\Gamma_k(E_k^V - E_k^M) - \tilde{\Gamma}_k k^2 H_k^M & -2(\tfrac{2}{5}\nu_k^V + \nu_k^M + \nu_k^R)\, k^2 E_k^V \\
 & & \qquad + O(a^2), \quad (3.1) \\
(\partial E_k^M/\partial t)_{\text{NLoc}} & = k\Gamma_k(E_k^V - E_k^M) + \tilde{\Gamma}_k H_k^V & + \alpha_k^R k^2 H_k^M - 2\nu_k^V k^2 E_k^M \\
 & & \qquad + O(a^2), \quad (3.2) \\
(\partial H_k^V/\partial t)_{\text{NLoc}} & = -k\Gamma_k(H_k^V - k^2 H_k^M) - \tilde{\Gamma}_k k^2 E_k^M & -2(\tfrac{2}{5}\nu_k^V + \nu_k^M + \nu_k^R)\, k^2 H_k^V \\
 & & \qquad + O(a^2), \quad (3.3) \\
(\partial H_k^M/\partial t)_{\text{NLoc}} & = (\Gamma_k/k)\,(H_k^V - k^2 H_k^M) + \tilde{\Gamma}_k E_k^V & + \alpha_k^R E_k^N - 2\nu_k^V k^2 H_k^M \\
 & & \qquad + O(a^2). \quad (3.4)
\end{array}
$$

† The eddy-turnover time for an eddy of scale $l$ and wavenumber $k = l^{-1}$ is defined as $l/v_l$, where

$$v_l = \left[2 \int_{l^{-1}}^{\infty} E_q^V \, dq\right]^{\frac{1}{2}}$$

is the $V$-energy per unit mass in scales smaller than $l$. In non-magnetic turbulence, the eddy-turnover time is also the time in which a sizable fraction of the energy in scales $\sim l$ is transferred to smaller scales.

The vertical line separates $LS$ contributions (on the left) from $SS$ contributions. The transport coefficients appearing in (3.1)–(3.4) are given by

$$\alpha_k^R = \alpha_k^V - \alpha_k^M, \tag{3.5}$$

$$\alpha_k^V = -\tfrac{4}{3}\int_{k/a}^{\infty} \theta_{kqq} H_q^V \, dq, \quad \alpha_k^M = -\tfrac{4}{3}\int_{k/a}^{\infty} \theta_{kqq} q^2 H_q^M \, dq, \tag{3.6}$$

$$\nu_k^V = \tfrac{2}{3}\int_{k/a}^{\infty} \theta_{kqq} E_q^V \, dq, \quad \nu_k^M = \tfrac{2}{3}\int_{k/a}^{\infty} \theta_{kqq} E_q^M \, dq, \tag{3.7}$$

$$\nu_k^R = \tfrac{2}{15}\int_{k/a}^{\infty} \theta_{kqq}^2 \, q \frac{\partial \mu_q}{\partial q} (E_q^V - E_q^M) \, dq, \tag{3.8}$$

$$\Gamma_k = \tfrac{4}{3} k \int_0^{ak} \theta_{kkq} E_q^M \, dq, \quad \tilde{\Gamma}_k = \tfrac{4}{3}\int_0^{ak} \theta_{kkq} q^2 H_q^M \, dq. \tag{3.9}$$

The $\Gamma$'s have the dimensions of velocity. Because of the realizability constraint $|H_q^M| \leqslant E_q^M/q$ (cf. equation (15) of I) we have

$$|\tilde{\Gamma}_k| \leqslant a\Gamma_k. \tag{3.10}$$

We propose to call $\alpha_k^V$, $\alpha_k^M$ and $\alpha_k^R$ kinetic, magnetic and residual *torsality* because they involve the torsion of $V$- and $M$-field lines. These very important transport coefficients, which have the dimensions of velocity, characterize the helicity or $\alpha$ effect. $\nu_k^V$, $\nu_k^M$ and $\nu_k^R$ will be called kinetic, magnetic and residual eddy diffusivities. Notice that $\nu_k^R$ is not positive-definite and vanishes at equipartition of $V$- and $M$-energy spectra. The $\alpha$'s like helicities, are pseudo-scalars, whereas the $\nu$'s are scalars.

*The Alfvén effect*

The lowest-order terms involving interactions with large scales are (a superscript $A$ stands for Alfvén)

$$(\partial E_k^V/\partial t)_{LS}^A = -k\Gamma_k(E_k^V - E_k^M), \tag{3.11}$$

$$(\partial E_k^M/\partial t)_{LS}^A = k\Gamma_k(E_k^V - E_k^M), \tag{3.12}$$

$$(\partial H_k^V/\partial t)_{LS}^A = -k\Gamma_k(H_k^V - k^2 H_k^M), \tag{3.13}$$

$$(\partial H_k^M/\partial t)_{LS}^A = (\Gamma_k/k)(H_k^V - k^2 H_k^M). \tag{3.14}$$

When the Alfvén contribution $\mu_k^A$ to the eddy-damping rate $\mu_k$ dominates the self-distortion and dissipation terms, it is easily checked that $\Gamma_k$ as given by (3.9) is essentially $b_0$, the r.m.s. $M$-field, which is also the typical group velocity of Alfvén waves. From (3.11)–(3.14) we find that under the action of random Alfvén waves the $V$- and $M$-energy spectra relax to equipartition in a time of the order of $(kb_0)^{-1}$, as predicted by Kraichnan (1965). Similarly, the helicity spectra relax to 'equipartition', which is now understood as

$$H_k^V = k^2 H_k^M, \tag{3.15}$$

the factor $k^2$ appearing because of the different dimensions of $V$- and $M$-helicity. Equivalently we can say that 'residual energy' $E_k^R = E_k^V - E_k^M$ and 'residual helicity' $H_k^R = H_k^V - k^2 H_k^M$ relax to zero. We call this the 'Alfvén effect'. The

$\tilde{\Gamma}_k$ terms in (3.1)–(3.4) are simply first-order corrections to the Alfvén effect involving the helicity of large-scale $M$-fields.

*The kinetic and magnetic helicity effects*

The lowest-order terms involving interactions with small-scale $V$- or $M$-helicity are (a superscript $H$ stands for helicity)

$$(\partial E_k^V/\partial t)_{SS}^H = 0, \quad (\partial H_k^V/\partial t)_{SS}^H = 0, \tag{3.16}$$

$$(\partial E_k^M/\partial t)_{SS}^H = \alpha_k^R\, k^2 H_k^M; \quad (\partial H_k^M/\partial t)_{SS}^H = \alpha_k^R\, E_k^M. \tag{3.17}$$

Notice that the residual torsality $\alpha_k^R$ is expressed in terms of the residual helicity $H_k^R = H_k^V - k^2 H_k^M$ of the small scales by

$$\alpha_k^R = -\tfrac{4}{3}\int_{k/a}^{\infty} dq\,\theta_{kqq} H_q^R. \tag{3.18}$$

When $\alpha_k^R$ is prescribed, (3.17) are easily integrated, yielding exponentially growing and decaying $M$-energy and $M$-helicity with a rate of growth (or decay) $k|\alpha_k^R|$. We may conclude that small-scale residual helicity destabilizes large-scale $M$-energy and $M$-helicity. This is reminiscent of the 'helicity or $\alpha$ effect' of Steenbeck *et al.* (1966) and Moffatt (1970*a*, *b*; see also Kraichnan 1976) with, however, an important change: both $V$- and $M$-helicities produce a destabilizing effect and it is the difference as measured by the residual helicity which is the true motor of the instability.

*Phenomenology of M-helicity effect*

Since the existence of an $M$-helicity effect does not seem to have been recognized earlier,† it is interesting to propose for it a simple phenomenological derivation. We assume the following situation at time $t_0$: no velocity field, a small-scale turbulent $M$-field $\mathbf{b}$ with $M$-helicity and a strong almost static large-scale random $M$-field $\mathbf{B}$ (when viewed from the small scales, $\mathbf{B}$ may be treated as uniform). Near $t_0$, the Lorentz force term will be the dominant one in the momentum equation:

$$\partial \mathbf{v}/\partial t = \mathbf{B}.\nabla\mathbf{b}. \tag{3.19}$$

Integrating with zero initial conditions on $\mathbf{v}$ we obtain

$$\mathbf{v}(t) \approx \mathbf{B}.\nabla\int_{t_0}^{t}\mathbf{b}(\tau)\,d\tau. \tag{3.20}$$

For an infinitely conducting medium we have the Ohm's law

$$\mathbf{E}(t) = -\mathbf{v}\wedge(\mathbf{b}+\mathbf{B}). \tag{3.21}$$

Assuming $\langle\mathbf{b}\rangle = 0$, using (3.20) and (3.21) and averaging over the small-scale randomness (operation denoted $\langle\,.\,\rangle_{SS}$) we obtain

$$\langle\mathbf{E}\rangle_{SS} = \int_{t_0}^{t}\langle\mathbf{b}(t)\wedge\mathbf{B}.\nabla\mathbf{b}(\tau)\rangle_{SS}\,d\tau. \tag{3.22}$$

† The magnetic gyrotropy effect introduced by Vainshtein (1972) and Vainshtein & Zeldovich (1972) is distinct from the magnetic helicity effect discussed here [see remark (5.1)]

Assuming isotropy this reduces to

$$\langle \mathbf{E} \rangle_{SS} = \tfrac{1}{3}\tau \langle \mathbf{b} . (\nabla \wedge \mathbf{b}) \rangle_{SS} \mathbf{B}, \tag{3.23}$$

where $\tau$ is a typical coherence time of the small-scale $M$-turbulence. The coefficient $\tau\langle \mathbf{b}.(\nabla \wedge \mathbf{b})\rangle_{SS}$ is essentially the magnetic torsality $\alpha_k^R$ defined by (3.5) for $k = 0$. When (3.23) is used in the induction equation $\partial \mathbf{B}/\partial t = -\nabla \wedge \mathbf{E}$ and averaging is performed also over the large scales, the well-known helicity-effect equation results:

$$\partial \langle \mathbf{B} \rangle / \partial t = -\alpha_0^M \nabla \wedge \langle \mathbf{B} \rangle. \tag{3.24}$$

This equation, when integrated for a Fourier mode of wavenumber $k$, gives a rate of growth of the mean $M$-field of $k|\alpha_0^M|$. If the large-scale $M$-field has zero mean, as assumed in this paper, a similar growth will be found for the large-scale $M$-energy. The above argument can also be given a purely geometric form in terms of twisting of $M$-field lines as is done for the ordinary (kinetic) helicity effect (Steenbeck *et al.* 1966). Notice that $V$-helicity will lead to (3.24) with $-\alpha_0^M$ replaced by $+\alpha_0^V$.

*The expression for the kinetic torsality*

Our expression (3.6) for the kinetic torsality resembles the one first given by Steenbeck *et al.* (1966):

$$\alpha^V = -\tfrac{1}{3}\tau \langle \mathbf{v} . \nabla \wedge \mathbf{v} \rangle. \tag{3.25}$$

If most of the kinetic helicity is in small scales with a typical eddy-damping time $\mu_q = \tau^{-1}$, then the two expressions are identical. However, our expression of the kinetic torsality $\alpha_k^V$ may be substantially smaller than that of Steenbeck *et al.* because the eddy-damping rate $\mu_q$ includes the Alfvén term

$$\sim q \left[ \int_0^q E_{q'}^M \, dq' \right]^{\frac{1}{2}}$$

(cf. § 2). As the large-scale $M$-energy grows, the kinetic torsality will be reduced. It is often considered that such a reduction is required to obtain a saturation mechanism (cf. review paper of Moffatt 1973); however, as we shall see in the next section, this probably is not the most effective saturation mechanism.

Another expression of the kinetic torsality is given by Moffatt (1970*a*) for the limiting case of low turbulent magnetic Reynolds number:

$$\alpha^V = -(2/3\lambda) \int_0^\infty q^{-2} H_q^V \, dq. \tag{3.26}$$

Our expression (3.6) will reduce to (3.26) only if most of the $V$-helicity is confined to the Joule dissipation range, where $\mu_q \sim \lambda q^2$. In strong MHD turbulence with reasonably high magnetic Reynolds number, the eddy-damping rate at helicity-containing wavenumbers is more likely to be independent of both viscosity and magnetic diffusivity; in this case, the difficulties of the limit $\lambda \to 0$ discussed by Moffatt (1974) do not arise.

*Kinetic and magnetic eddy diffusivities*

In the non-local equations (3.1)–(3.4) the terms involving the $\nu$'s, which are all of first order, describe the diffusive action of small-scale turbulence on large scales. It is interesting to note that magnetic diffusivity is present only in the equations for kinetic quantities. This result may be recovered in an elementary way: let there be given initially at $t_0$ a uniform current $\mathbf{J}$ imbedded in small-scale $M$-turbulence. To lowest order, we have for the small-scale velocity

$$\partial \mathbf{v}/\partial t = \mathbf{J} \wedge \mathbf{b} - \nabla p = -(\mathbf{J}.\nabla)\,\mathbf{a}, \tag{3.27}$$

where $\mathbf{a}$ is the vector potential. Integrating and using Ohm's law, we obtain

$$\mathbf{E}(t) = \int_{t_0}^{t} [(\mathbf{J}.\nabla)\,\mathbf{a}(\tau) \wedge \mathbf{b}(t)]\,d\tau. \tag{3.28}$$

It is easily checked that this electric field is irrotational and thus does not contribute to the induction term, so that the $M$-field is unaffected.

*Numerical treatment of non-local interactions*

It is known that conservation of total energy and of $M$-helicity holds not only for the complete nonlinear interaction but also 'in detail' for each triad of interacting wavenumbers (cf. I). The question therefore arises whether conservation is satisfied to each order of the non-local expansion. It may be checked in (3.4) that $M$-helicity is not conserved by the zeroth-order Alfvén and helicity terms separately. However, after combination, we obtain (Nloc stands for all non-local terms)

$$(\partial H_k^M/\partial t)_{\mathrm{NLoc}} = (\Gamma_k/k)\,(H_k^V - k^2 H_k^M) + \alpha_k^R E_k^M, \tag{3.29}$$

which integrates to zero over the wavenumber $k$ and also gives zero after summation over all terms involving a given triad $(k, p, q)$ (detailed conservation). Similar conservation takes place to first order. The situation is different for total energy conservation: the zeroth-order Alfvén terms give conservation but the zeroth-order helicity terms must be combined with the first-order Alfvén terms (involving $\tilde{\Gamma}_k$) to ensure conservation. Therefore, if we want a numerical approximation of the non-local interactions based on the expansions (3.1)–(3.4) and which has all the desired conservation properties, we cannot keep only the zeroth-order terms which are physically the most important ones. The lowest-order conservation-consistent system of non-local equations which we have used for numerical purposes retains all the terms in (3.1)–(3.4) with the exception of the eddy-diffusivity terms and the helical corrections to the Alfvén effect in the rate of change of the kinetic and magnetic helicities. This system is integrated numerically together with the local contributions to the full spectral equations of table 1 (cf. § 2). We have taken three points per octave ($F = 3$), giving $a = 0{\cdot}26$ for the expansion parameter.

## 4. The $-\frac{3}{2}$ inertial range of non-helical MHD turbulence

The non-helical problem has already been considered by Kraichnan (1965), Kraichnan & Nagarajan (1967), Nagarajan (1971), Orszag (1975) and others. The existence for strong MHD turbulence of a $-\frac{3}{2}$ inertial range with equipartition of $V$- and $M$-energy predicted by Kraichnan (1965) on the basis of phenomenological arguments has never been demonstrated either theoretically or numerically. This question will now be investigated on the basis of the EDQNM spectral equations given in table 1 with all the helical terms removed.

Let us consider fully developed stationary MHD turbulence maintained by forcing near the wavenumber $k_E$ with an injection rate $\epsilon$ of total (kinetic plus magnetic) energy. Most of the energy will then be in the energy range. We assume the existence of some inertial range of the form

$$E_k^V = C_V k^{-n_V}, \quad E_k^M = C_M k^{-n_M}, \tag{4.1}$$

with $n_V$ and $n_M$ greater than one to ensure finite total energy. It is then easily checked that, for large enough $k$, the eddy-damping rate $\mu_k$ reduces to the Alfvén contribution

$$\mu_k = \mu_k^A = \frac{k}{\sqrt{3}}\left[2\int_0^k E_q^M\, dq\right]^{\frac{1}{2}} \approx \frac{kb_0}{\sqrt{3}}. \tag{4.2}$$

At this point we make a brief digression to recall Kraichnan's (1965) phenomenological argument. Since the triad-relaxation time $\theta_{kpq}$ for wavenumbers triads $\sim k$ is now $\sim (kb_0)^{-1}$, the energy transfer rate $\epsilon$ is expected to be inversely proportional to $b_0$. Dimensional analysis then gives an unambiguous expression for $\epsilon$ in terms of $b_0$, $k$ and $E_k = E_k^V = E_k^M$:

$$\epsilon \sim b_0^{-1} k^3 (E_k)^2,$$

whence

$$E_k = C(\epsilon b_0)^{\frac{1}{2}} k^{-\frac{3}{2}}, \tag{4.3}$$

where $C$ is a dimensionless constant. We show now that this argument can be made rigorous. For this, we define the energy flux as the rate of transfer of total energy through the wavenumber $K$ by (cf. Kraichnan 1959)

$$\Pi_K^T = -\left(\frac{\partial}{\partial t}\right)_{\text{NLin}} \int_0^K (E_k^V + E_k^N)\, dk, \tag{4.4}$$

where $(\partial/\partial t)_{\text{NLin}}$ means the contribution from the nonlinear (transfer) terms, excluding dissipation and forcing. In order to have an inertial range, we must ensure the constancy of the energy flux over a wavenumber range $k_E \ll k \ll k_D$, where $k_D$ is a dissipation wavenumber, which can be arbitrarily large. Using (4.1) and (4.2) in (4.4) we find that the constancy of $\Pi_K^T$ requires as a necessary condition $n_V = n_M = \frac{3}{2}$. It remains to investigate the convergence of the integrals. Using the expression for the geometric coefficients given in the appendix, we find that the energy flux diverges like

$$(C_M - C_V)\int_0^K q^{-\frac{3}{2}}\, dq,$$

the divergence being removed when $C_M = C_V$, i.e. at equipartition.

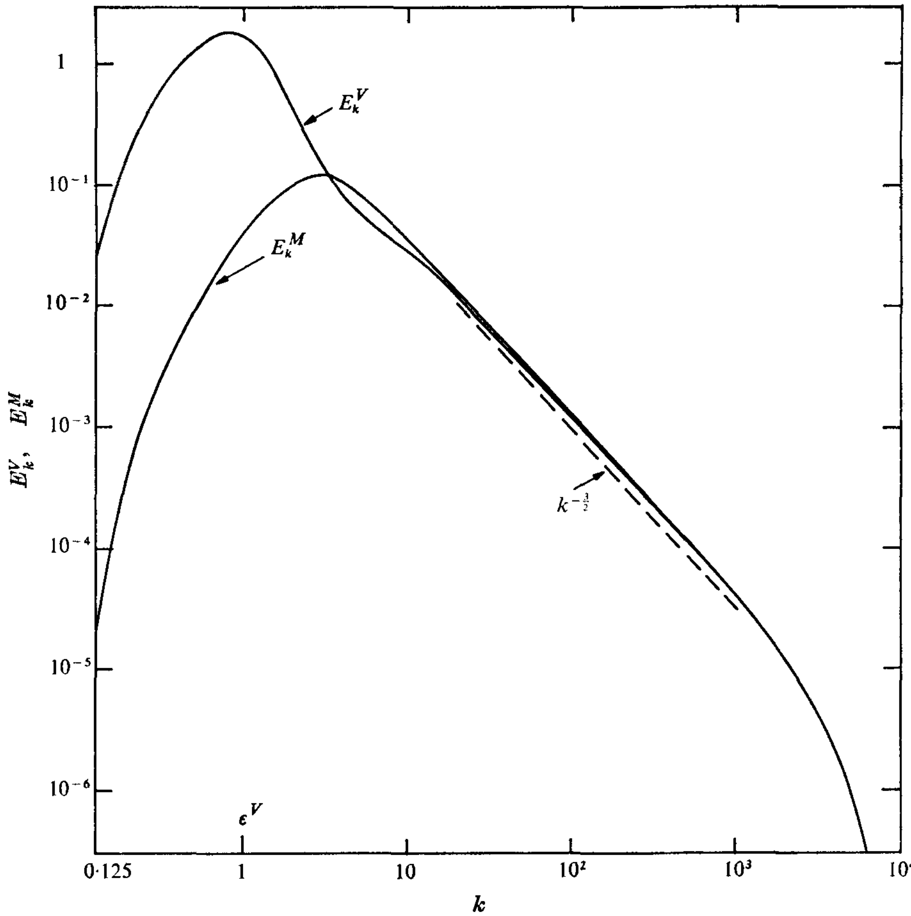


FIGURE 1. The $-\frac{3}{2}$ inertial range of non-helical MHD turbulence. No helicity. $V$-energy injection spectrum $F_k^V = \frac{1}{2}F_k$, where $F_k = Ck^4 \exp(-2k^2)$ $\left(C \text{ chosen to give } \int_0^\infty F_k\, dk = 1\right)$. Initial conditions: $E_k^V = 0$, $E_k^M = 10^{-3}F_k$. Minimum and maximum wavenumbers: $2^{-3}$ and $2^{14}$. Magnetic Prandtl number unity, $\nu = \lambda = 10^{-6}$. $V$- and $M$-energy spectra represented at $t = 12$ (large-eddy-turnover time of order unity). Notice the slight excess of $M$-energy in the inertial range.

The constancy of the flux of total energy is not sufficient by itself to ensure the existence of a stationary inertial range. Indeed, it only implies zero transfer of the total energy, leaving open the possibility of transfer from $V$- to $M$-energy or vice versa. In fact, if (4.1) and (4.2) (with $C_M = C_V$ and $n_V = n_M = \frac{3}{2}$) are used in the expression for the $V$- and $M$-transfer functions, given in table 1, it is found that $M$-transfer is positive and hence $V$-transfer is negative. To obtain identically zero transfer we must correct the equipartition $-\frac{3}{2}$ range by higher-order terms. We found that the first correction follows a $-2$ power law with an excess of $M$-energy; higher-order corrections have not been investigated. Notice that, in spite of the corrections, the relative excess of $M$-energy

$$(E_k^M - E_k^V)/(E_k^M + E_k^V)$$

goes to zero as $k \to \infty$.

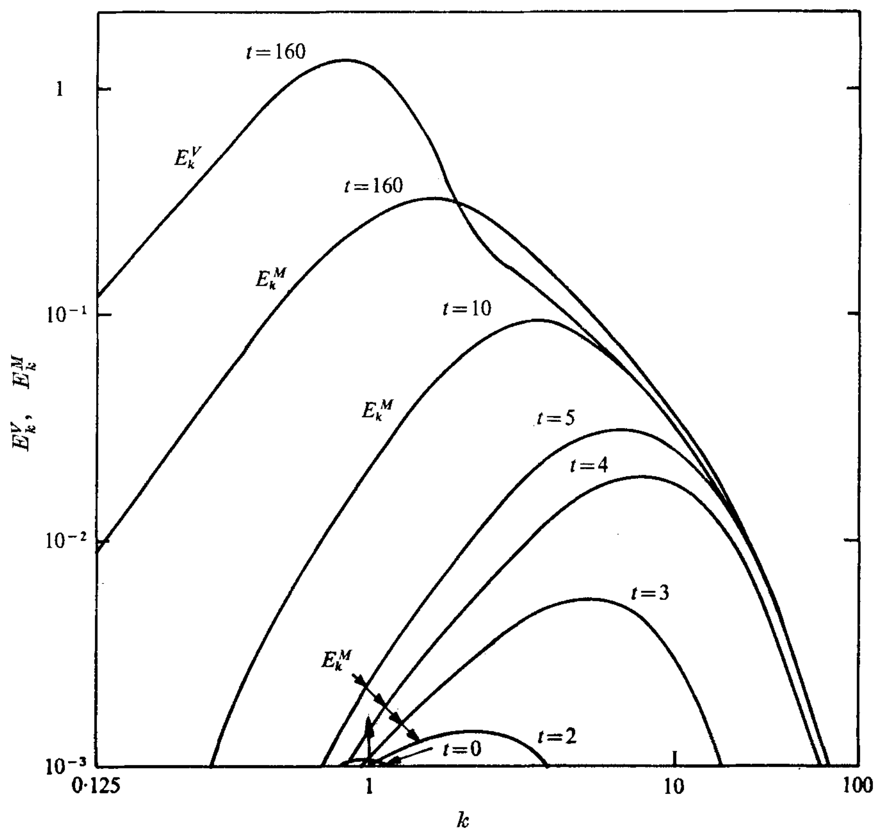


FIGURE 2. Growth to equipartition of seed of $M$-energy. No helicity. Only $V$-energy injection $F_k^V = F_k$ (defined in figure 1). Initial conditions: $E_k^V(0) = 0$ and $E_k^M(0) = 10^{-3}F_k$. Minimum and maximum wavenumbers: $2^{-3}$ and $2^7$. Magnetic Prandtl number unity, $\nu = \lambda = \frac{1}{300}$. Evolution of $V$- and $M$-energy spectra.

As an illustration of our analysis we have integrated numerically the non-helical spectral equations. To specify the data we introduce the forcing function

$$F_k = Ck^4 \exp(-2k^2), \tag{4.5}$$

where $C$ is chosen such that $F_k$ integrates to unity. With this choice the forcing wavenumber is $k_E = 1$. Only $V$-energy is injected with a forcing spectrum $F_k^V = F_k$. The magnetic Prandtl number is unity, $\lambda = \nu = 10^{-6}$; minimum and maximum wavenumbers are $2^{-3}$ and $2^{14}$; the initial conditions are zero for $E_k^V$ and $10^{-3}F_k$ for $E_k^M$. After a few large-eddy turnover times (here of order unity), a stationary state is reached in the small scales, displayed in figure 1, which exhibits clearly a $-\frac{3}{2}$ inertial range with a slight excess of $M$-energy.

Many more aspects of non-helical MHD turbulence may be investigated with the EDQNM spectral equations. These include, for example, the question of dependence on the magnetic Prandtl number, which is of great importance in astrophysical applications. Since the main purpose of this paper is to study helical turbulence, we shall leave out such problems.

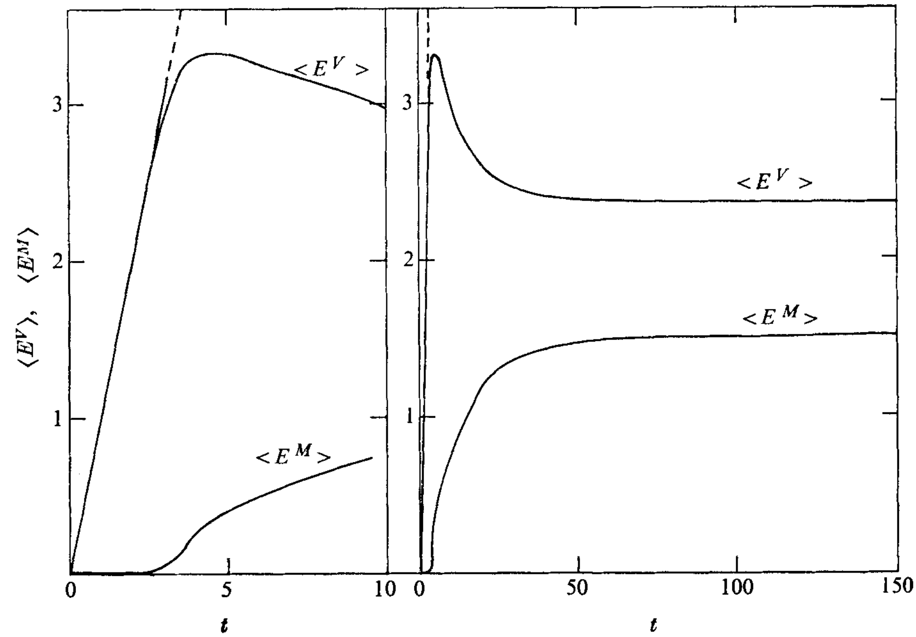


FIGURE 3. Evolution of mean $V$ - and $M$-energies. $E_k^V(0) = 2{\cdot}8\, F_k$. Conditions same as in figure 2. The behaviour for times of the order of the large-eddy-turnover time is shown with a dilated time scale on the left.

Let us now briefly consider a question which is of importance in the dynamo problem: does an initial seed of magnetic energy grow to equipartition at large wavenumbers, a possibility first discussed by Batchelor (1950)? This problem is considered in Kraichnan & Nagarajan (1967) in its linear version (turbulent $V$-field prescribed). It is argued there that growth is not automatically expected since the rate of stretching of field lines is of the same order as the rate of turbulent dissipation (magnetic excitation being carried along the cascade to the dissipative sink at high wavenumbers). We have investigated this problem numerically with the full nonlinear spectral equations (non-helical). The results are displayed in figure 2, which shows the evolution of the $V$- and $M$-spectra. Only $V$-energy is injected using $F_k^V = F_k$, where $F_k$ is defined in (4.5). The initial spectra are $E_k^V(0) = 0$ and $E_k^M(0) = 10^{-3} F_k$. The magnetic Prandtl number is unity, $\nu = \lambda = \frac{1}{300}$. Notice the growth of the $M$-energy, first at the highest wavenumbers, where equipartition is obtained; the bottom of the equipartition range then moves to smaller wavenumbers in a way reminiscent of the inverse cascade of errors in the predictability problem of ordinary turbulence (Leith & Kraichnan 1972). Approximate equipartition is obtained at small scales in a few eddy-turnover times. On the large scales more time is needed to reach stationarity; this is seen in figure 3, which shows, under the same conditions as figure 2, the evolution of the total $V$- and $M$-energies, which reach, after about fifty large-eddy turnover times, saturation values of 2·4 and 1·5. It must finally be emphasized that, if the Reynolds number is very large, the growth of the $M$-field cannot be correctly analysed solely with the linear Ohm's law even when the mean

$M$-energy is only a small fraction of the mean $V$-energy. The reason is that the $M$-energy, however small, will give rise to Alfvén waves (not describable in terms of the Ohm's law) on sufficiently small scales, which then will rapidly bring $V$- and $M$-excitation to equipartition. Of course, if $b_0$ goes to zero, the Reynolds number above which the linear treatment becomes invalid goes to infinity.

## 5. The inverse cascade of magnetic helicity

It was conjectured in I that the presence of $M$-helicity introduces a very important novel feature into three-dimensional MHD turbulence, namely the existence of an inverse cascade of $M$-helicity from small to large scales; thus far only two-dimensional non-magnetic turbulence was known to possess an inverse cascade.

*Initial transfer*

Let us consider the simplest helical initial-value problem with no forcing. We take

$$\left.\begin{aligned} E_k^V(0) = E_k^M(0) = F_k, \\ H_k^V(0) = 0, \quad H_k^M(0) = \xi F_k/k, \end{aligned}\right\} \tag{5.1}$$

where the initial energy spectrum $F_k$ is given. $\xi$ measures the relative $M$-helicity at the initial time. We assume that most of the initial energy lies near the wavenumber $k_E$ (if $F_k$ is given by (4.5), $k_E = 1$). The question is: is there initially any transfer of magnetic energy to small wavenumbers $k \ll k_E$? If the initial conditions are taken to be Gaussian, the initial triple correlations and the initial time derivatives of the spectra vanish. Let us then evaluate the second derivative of the $M$-energy spectrum at time $t = 0$ for wavenumbers $k \ll k_E$. Using (3.2) and (3.9) and the fact that $\theta_{kpq}(t) = t + O(t^2)$ [cf. (2.4)] we obtain

$$(\partial^2 E_k^M/\partial t^2)_{t=0} \approx \tfrac{4}{3}\xi^2 k F_k \int_{q \gg k} dq\, q F_q > 0, \tag{5.2}$$

where most of the integral comes from $q \sim k_E$. Since the right-hand side of (5.2) is clearly positive, we see that the presence of $M$-helicity induces an initial transfer of $M$-energy to small wavenumbers, however small the relative amount of $M$-helicity $\xi$. It must be stressed that, when Gaussian initial conditions are used, the full MHD equations and the EDQNM approximation agree to $O(t^3)$; therefore (5.2) holds also for the full MHD equations. The initial positive transfer of $M$-energy at small wavenumbers is indeed observed in the direct numerical simulations of Pouquet & Patterson (1976).†

*Numerical integration*

To discover whether inverse transfer is a permanent feature of helical MHD turbulence, it is necessary to inject $M$-helicity, say near $k_E$, and see whether

† In the non-helical case, the corresponding initial-value problem leads to a negative second derivative of the $M$-energy spectrum for small wavenumbers stemming from the eddy-diffusivity term in (3.2).

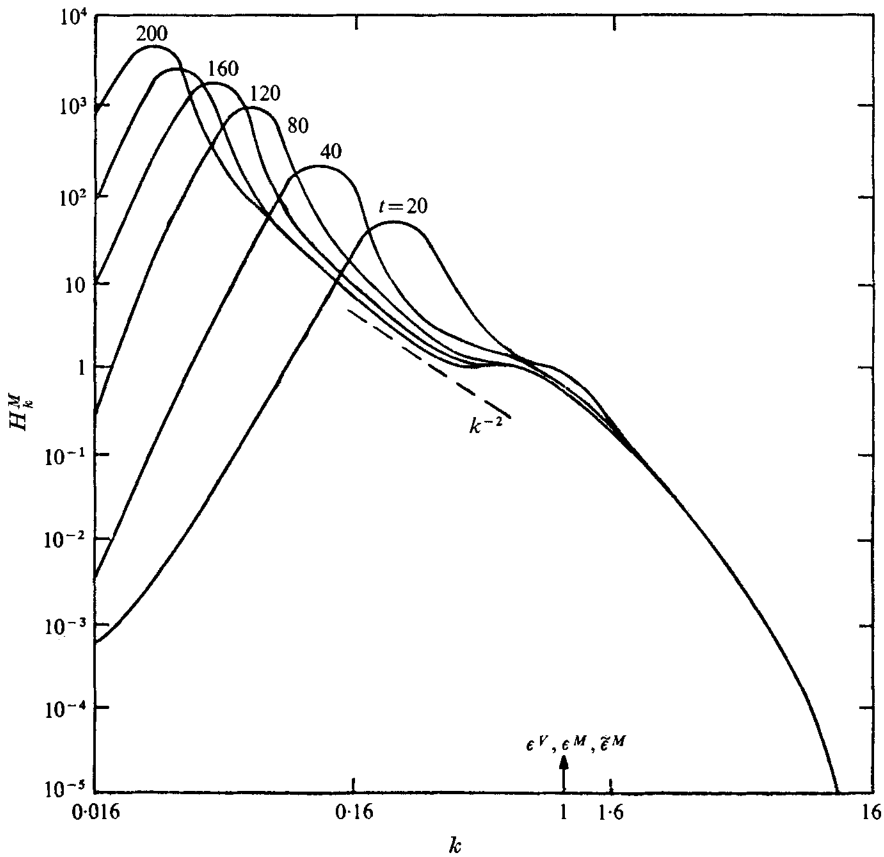


FIGURE 4. The inverse cascade of $M$-helicity. Injection of $V$- and $M$-energy and $M$-helicity (maximal): $F_k^V = F_k^M = k\tilde{F}_k^M = \frac{1}{2}F_k$ ($F_k$ defined in figure 1), $\tilde{F}_k^V = 0$. Initial conditions: zero. Minimum and maximum wavenumbers: $2^{-6}$ and $2^4$. Magnetic Prandtl number unity, $\nu = \lambda = \frac{1}{30}$. Evolution of $M$-helicity spectrum for times large compared with the eddy-turnover time at the injection wavenumber.

an inverse cascade builds up. This is easily done numerically using the spectral equations. The energy injection spectra are taken to be as in the non-helical case of § 4:

$$F_k^V = F_k^M = 0{\cdot}5F_k,$$

where $F_k$ is defined by (4.5). The helicity injection spectra are

$$\tilde{F}_k^V = 0, \quad \tilde{F}_k^M = 0{\cdot}5F_k/k. \tag{5.3}$$

This choice corresponds to the maximal injection rate of $M$-helicity (maximality is not required but saves computer time). The initial conditions are zero; the minimum and maximum wavenumbers are $2^{-6}$ and $2^4$. The magnetic Prandtl number is unity: $\nu = \lambda = \frac{1}{30}$. In figure 4 the $M$-helicity spectrum is plotted at different times large compared with the eddy-turnover time at the injection wavenumber, the latter being of order unity. As time progresses, more and more $M$-helicity is transferred to small wavenumbers. For a fixed wavenumber

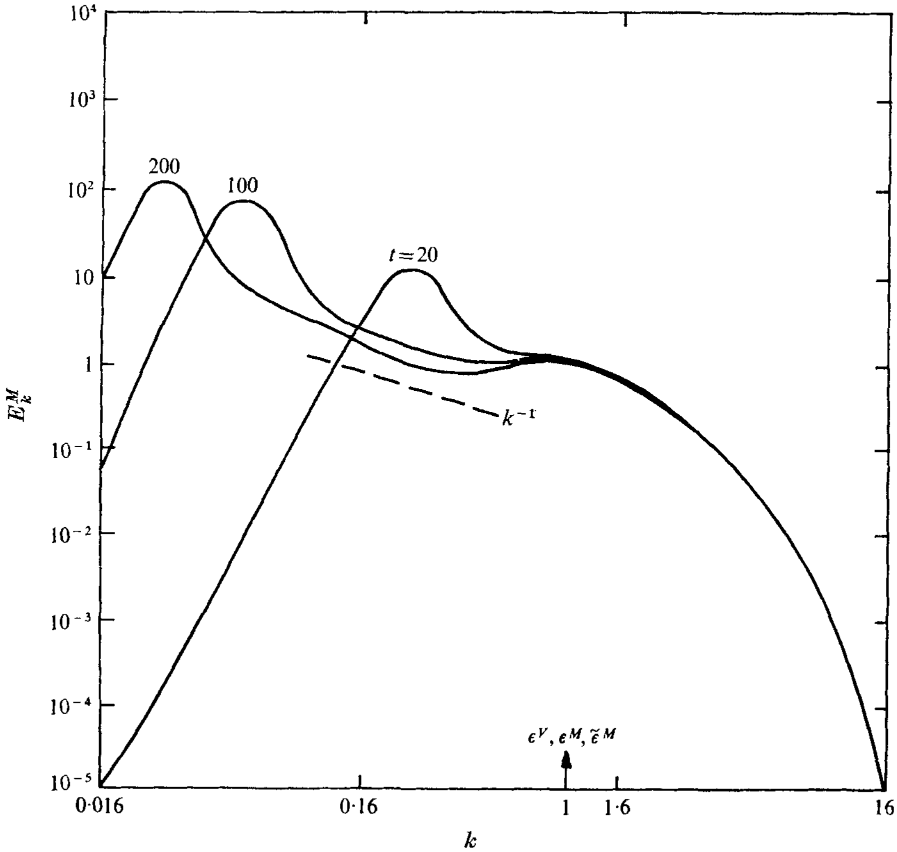


FIGURE 5. Variation of $M$-energy spectrum in the inverse cascade. Same conditions as in figure 4.

$k$ ($k_{\min} < k < k_E$), $H_k^M(t)$ converges to a quasi-stationary spectrum which follows approximately a $-2$ law. For fixed $t$, $H_k^M$ achieves its maximum at a wavenumber $k_1(t) \sim t^{-1}$. The $M$-energy spectrum is displayed in figure 5. It has the same overall features as the $M$-helicity spectrum and follows approximately a $-1$ law. The $V$-energy spectrum at $t = 200$ is displayed together with the $M$-energy spectrum in figure 6. Approximate equipartition is obtained except at the smallest wavenumbers. Similar results hold for $V$- and $M$-helicities. The variation of the total $M$-helicity and energy are plotted in figure 7 together with the total injected helicity and energy; notice that $M$-helicity increases almost linearly with a slope slightly less than the injected helicity, an indication that most of the injected $M$-helicity is transferred to small wavenumbers rather than being dissipated. This is in contrast with the total mean $M$-energy, which increases much more slowly and may (possibly) saturate; still, the $M$-energy reaches at $t = 200$ a value more than twice the equipartition value attained in a similar non-helical run. Computer CPU time up to $t = 200$ was about 20 min on a CDC 7600.

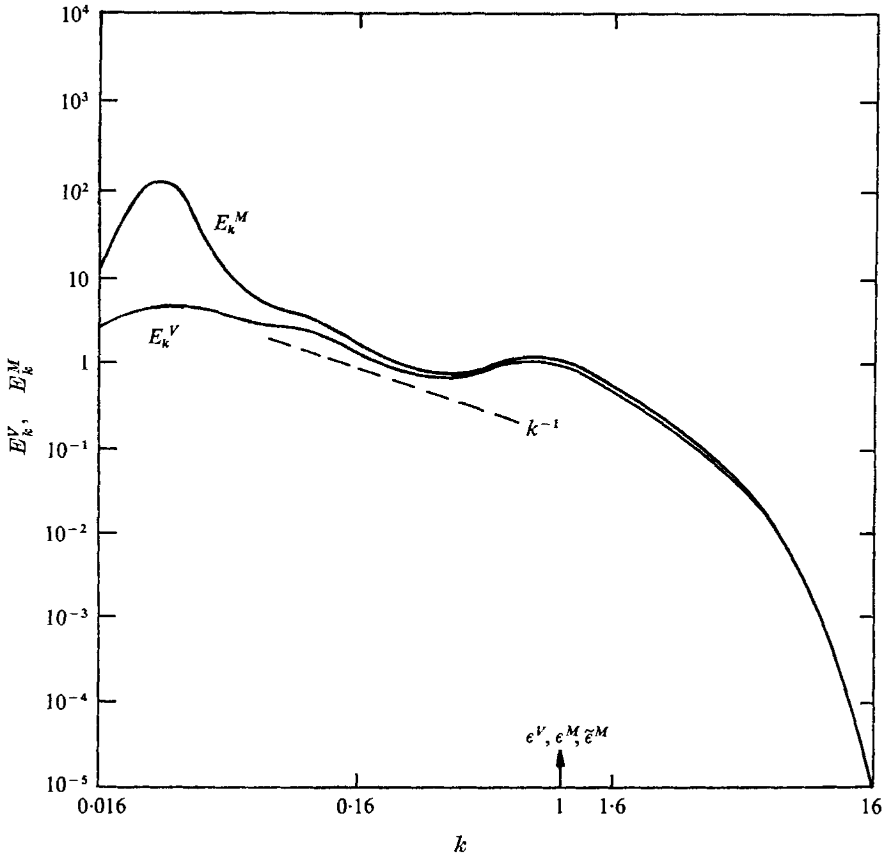


FIGURE 6. $V$- and $M$-energy spectra in the inverse cascade at $t = 200$. Same conditions as in figure 4.

*The inverse cascade as a competition between Alfvén and helicity effects*

The physics of the inverse cascade of $M$-helicity can be readily understood in terms of the two basic non-local effects introduced in § 3. The residual (kinetic minus magnetic) helicity in the energy range, say $k \sim k_E$, coming from the $M$-helicity injection, produces, by the helicity effect, a growth of both $M$-energy and $M$-helicity in smaller wavenumbers, say $k \sim \frac{1}{2}k_E$; the growing $M$-energy near $\frac{1}{2}k_E$ tends by the Alfvén effect to reduce the residual helicity near $k_E$ whereas the growing $M$-helicity near $\frac{1}{2}k_E$ destabilizes smaller wavenumbers, say $k \sim \frac{1}{2}k_E$, and so on. The appearance of $V$-energy and helicity also in the small wavenumbers can be explained by the action of the Lorentz force. Notice that, by the above mechanism, saturation of the spectra at a given wavenumber is obtained but not overall saturation since the inverse cascade may proceed to ever-smaller wavenumbers.

*Remark* (5.1) (concerning the saturation mechanism of Vainshtein & Zeldovich). Another saturation mechanism which should prevent indefinite growth of large-scale $M$-fields is invoked by Vainshtein (1927) and Vainshtein & Zeldovich

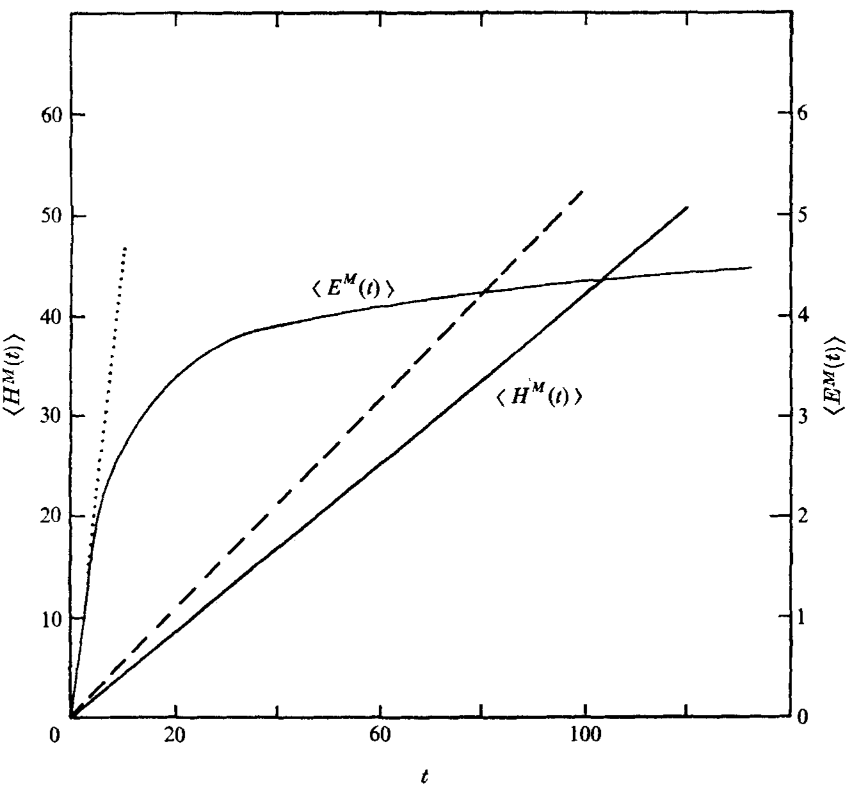


FIGURE 7. Mean $M$-energy and helicity in the inverse cascade. Same conditions as in figure 4. Dotted and dashed lines are total injected $M$-energy and helicity.

(1972). In our language their argument may be analysed as follows. Let there be given initially small-scale positive $V$-helicity $H_k^V > 0$ and (implicitly) no $M$-helicity; then $\alpha_k^R$ will be negative and by (3.17) the large-scale $M$-helicity will grow, at a rate $k|\alpha_k^R|$, to large negative values, whence the helical correction coefficient to the Alfvén effect $\tilde{\Gamma}_k$ given by (3.9) will be negative and because of (3.3) small-scale $V$-helicity will be *increased*. Now, the conclusion of Vainshtein & Zeldovich is that, under the same conditions, small-scale $V$-helicity is *reduced* as needed for saturation. The 'negative' $V$-helicity so generated is what they call 'magnetic gyrotropy'. Even if the effect had the sign claimed, it would still be much smaller than the saturation effect resulting from the simultaneous consideration of $V$- and $M$-helicity described in § 5.

### *The* $-2$ *inertial range of M-helicity*

Let us apply a simple dimensional argument of the Kolmogorov type to the inverse cascade of $M$-helicity. Denoting by $\tilde{\epsilon}_{tr}^M$ the absolute value of the rate of transfer of $M$-helicity in the inverse cascade (which may differ from the injection rate $\tilde{\epsilon}^V$ if some $M$-helicity is lost through dissipation), we assert that $E_k^M$ and

$H_k^M$ are functions solely of $\tilde{\epsilon}_{tr}^M$ and $k$, to obtain (with dimensionless constants $C_1$ and $C_2$)

$$H_k^M = C_2(\tilde{\epsilon}_{tr}^M)^{\frac{2}{3}}\, k^{-2}, \quad E_k^M = C_1(\tilde{\epsilon}_{tr}^M)^{\frac{2}{3}}\, k^{-1}, \tag{5.4}$$

which is in reasonable agreement with our numerical results for the quasi-stationary spectra. This type of reasoning must however be used with great care since it is known that the result may be altered by non-local effects: in the non-helical case of § 4, naïve dimensional reasoning would have produced a $-\frac{5}{3}$ instead of a $-\frac{3}{2}$ inertial range. A deductive theory of the inertial-range spectra starting from the fundamental MHD equations is ruled out in the present state of turbulence theory. However, such questions can usually be answered completely within the framework of spectral equations as was done for the $-\frac{3}{2}$ non-helical inertial range. For helical turbulence, the idea would be to introduce power-law spectra into the analytic expressions for the $M$-helicity flux $\tilde{\Pi}_k^M$, defined as in (4.4) with $H_k^M$ instead of $E_k^V + E_k^M$; then one imposes

$$\lim_{k\to 0} \tilde{\Pi}_k^M = -\tilde{\epsilon}_{tr}^M,$$

which expresses the (asymptotic) constancy of $M$-helicity flux. We have found that this constancy cannot be achieved if the inertial-range exponents differ from those given by (5.4). However, with this choice, $\tilde{\Pi}_k^M$ still has a logarithmic dependence on $k$. This is reminiscent of what happens in the enstrophy inertial range of two-dimensional non-magnetic turbulence (Kraichnan 1971*b*). In the two-dimensional case, the log dependence originates from non-local effects and can be eliminated by a suitable log correction to the spectrum itself. We have tried a similar procedure in the present case but have encountered certain difficulties which have not been overcome. Whether or not there is a log correction in the $M$-energy and the $M$-helicity spectra may seem an academic question which is certainly beyond direct numerical and experimental verification. However, connected with this, there are some important unsettled questions.

(i) For a $-1$ quasi-stationary $M$-energy spectrum, the mean $M$-energy diverges logarithmically at $k = 0$; a log correction may possibly imply a *saturation* of the mean $M$-energy.

(ii) Will exact equipartition of energies and helicities be achieved in the quasi-stationary solution? This question is connected with the previous one, since the more $M$-energy present in the small wavenumbers, the more $V$- and $M$-excitations will be coupled by the Alfvén effect.

*Remark* (5.2). A possibility which cannot be ruled out is that the eddy-damping rate given by (2.9), which has a positive Alfvén contribution, should also have a negative helicity contribution to account for the destabilization of large-scale triple correlations by small-scale helicity. This can however make the total eddy-damping rate negative, a situation which probably no Markovian theory such as the EDQNM or the TFM can meet. The direct-interaction approximation (Kraichnan 1958, 1959), in spite of its statistical Galilean non-invariance, may possibly be more adequate.

## 6. The nonlinear turbulent dynamo

We come now to the fundamental question of this work: can large-scale $M$-fields be generated by turbulence? We already know from § 4 that in the non-helical case an initial seed of $M$-energy will grow to equipartition values at small scales in a few local eddy-turnover times. We also know from § 5 that, if *magnetic* helicity is injected, large-scale $M$-energy will appear as a consequence of the inverse cascade. But what happens if we make the more realistic assumption that only *kinetic* helicity is being injected, along with $V$-energy? Since $V$-helicity is not conserved in MHD by the nonlinear interactions a cascade (direct or inverse) of $V$-helicity is not expected. Nevertheless, we shall see that an inverse cascade of $M$-helicity still takes place in spite of the absence of injection of $M$-helicity. This may be understood as follows.

Let us consider a purely kinetic turbulence maintained by $V$-energy and positive $V$-helicity injection near a wavenumber $k_E$; this is the non-magnetic helical case considered by Brissaud *et al.* (1973) and André & Lesieur (1976). Now let us introduce an initial weak seed of $M$-field at time $t_0$. After a few local eddy-turnover times, the $M$-energy in high wavenumbers will have grown to equipartition values with $V$-energy and we then have a truly MHD situation. The $M$-energy interacting with $V$-helicity will then generate $M$-helicity through the term $T^{\tilde{M}}_{\tilde{V}M}$ of the spectral equation (cf. table 1); it may be checked that this $M$-helicity is positive at large wavenumbers and negative at small wavenumbers. First the generated spectrum will integrate to zero since the nonlinear interactions conserve total $M$-helicity; however, the positive $M$-helicity at large wavenumbers will be removed by dissipation, which acts like a source of negative $M$-helicity. This negative $M$-helicity will now cascade to small wavenumbers, following the scheme described in the previous section. Calling $\tilde{\epsilon}^V$ the (positive) injection rate of $V$-helicity and $\tilde{\epsilon}^M_{\mathrm{tr}}$ the resulting (negative) rate of transfer of $M$-helicity to small wavenumbers, a dimensional argument indicates that

$$\tilde{\epsilon}^M_{\mathrm{tr}} = -\tilde{\epsilon}^V k_*^{-2}, \tag{6.1}$$

where $k_*$ is a typical wavenumber for generation of $M$-helicity. Since most of the excitation is in the energy range, we expect to have

$$k_* \approx k_E. \tag{6.2}$$

The above ideas are now checked numerically. For this, the injection spectra are

$$F^V_k = 0{\cdot}5F_k, \quad \tilde{F}^V_k = kF^V_k, \quad F^M_k = 0, \quad \tilde{F}^M_k = 0, \tag{6.3}$$

where $F_k$ is given as usual by (4.5); notice that the choice of $\tilde{F}^V_k$ corresponds to maximal $V$-helicity injection (cf. I). The $V$-helicity injection rate is $\tilde{\epsilon}^V = 0{\cdot}532$. The initial conditions are

$$E^V_k(0) = F_k, \quad H^V_k(0) = 0, \quad E^M_k(0) = 0{\cdot}1F_k, \quad H^M_k(0) = 0.$$

Except for short times, the value of $E^V_k(0)$ is irrelevant because of the injection of energy; the same holds for $E^M_k(0)$ as long as it is not identically zero. The

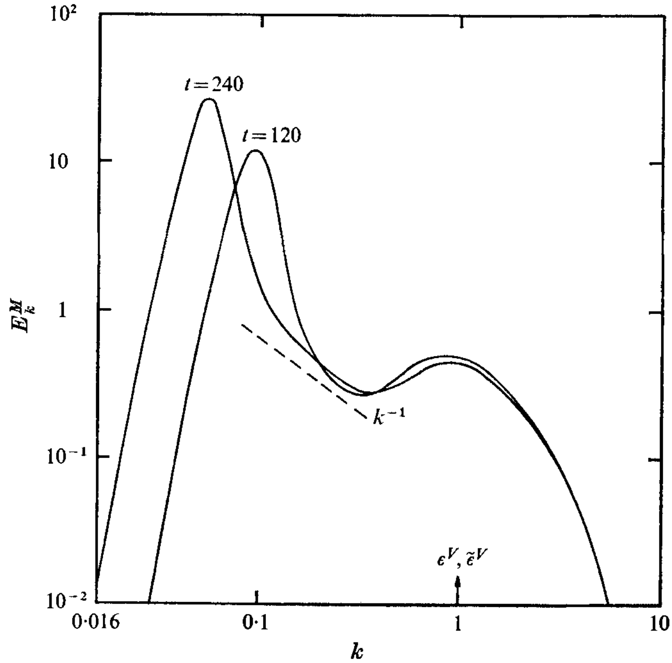


FIGURE 8. $M$-energy spectrum in the turbulent dynamo. Injection of only $V$-energy and helicity (maximal): $F_k^V = \tilde{F}_k^V/k = \frac{1}{2}F_k$. ($F_k$ defined in figure 1), $F_k^M = \tilde{F}_k^M = 0$. Initial conditions (seed of $M$-energy): $E_k^V(0) = F_k, H_k^V(0) = 0, E_k^M(0) = 0{\cdot}1\, F_k$ and $H_k^M(0) = 0$. Minimum and maximum wavenumbers: $2^{-6}$ and $2^4$. Magnetic Prandtl number unity, $\nu = \lambda = \frac{1}{30}$. Notice the build-up of large scale $M$-energy.

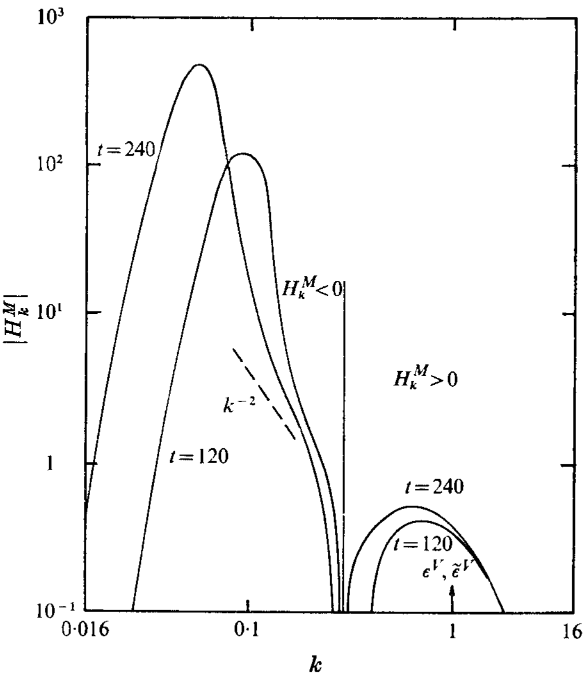


FIGURE 9. $M$-helicity spectrum in the turbulent dynamo. Same conditions as in figure 8. The $M$-helicity spectrum is positive to the right of the vertical line and negative to the left.

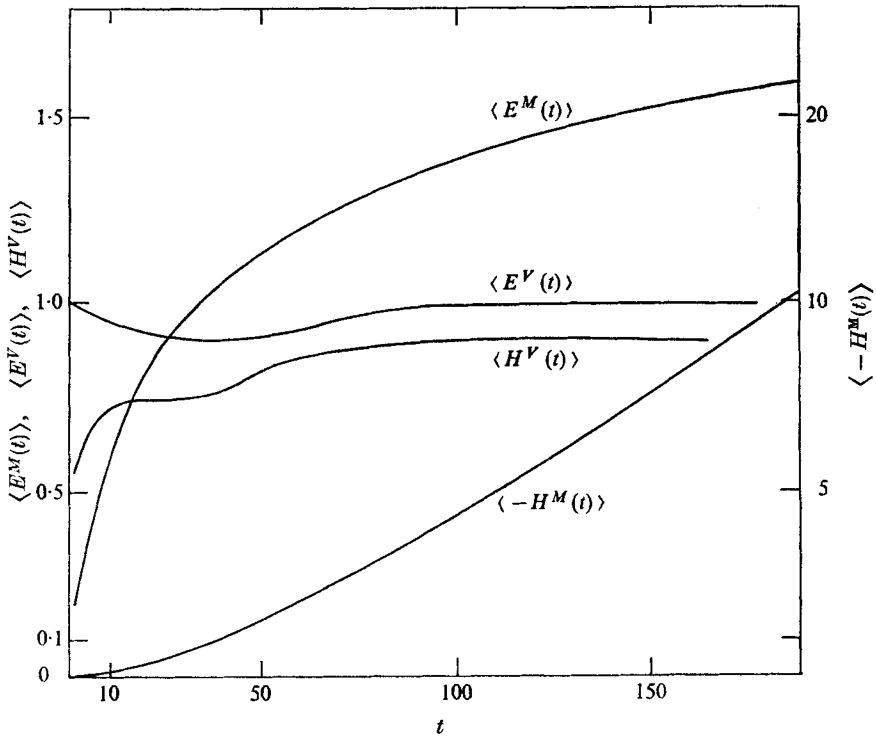


FIGURE 10. Mean $V$- and $M$-energies and helicities in the turbulent dynamo. Same conditions as in figure 8. Notice the saturation of $V$-energy and helicity, but not of $M$-energy, and the linear growth of $M$-helicity.

minimum and maximum wavenumbers are $2^{-6}$ and $2^4$. The magnetic Prandtl number is unity: $\nu = \lambda = \frac{1}{30}$. The eddy-turnover time at the injection wavenumber is of order unity and the full helical equations have been integrated up to $t = 240$. The $M$-energy and helicity spectra at $t = 120$ and $t = 240$ are shown in figures 8 and 9. The appearance of large-scale $M$-energy and helicity is particularly conspicuous although the $-1$ inertial range of $M$-energy and, even more, the $-2$ inertial range of $M$-helicity are not very well defined. The evolution of the mean $M$- and $V$-energies and helicities is shown in figure 10. Notice that $V$-energy and helicity remain practically constant after a few large-eddy turnover times whereas $M$-helicity, which starts from zero, has an asymptotically linear growth with a rate $\tilde{\epsilon}^M = -0{\cdot}065$, corresponding to $k_* = 2{\cdot}9k_E$ in (6.2); $M$-energy exceeds $V$-energy for $t \gtrsim 30$ and grows approximately like $t^{\frac{1}{2}}$. Up to $t = 30$ the yield of the dynamo, defined as mean $M$-energy divided by the total injected $V$-energy, is of the order of 7 %.

It may be of interest to estimate the order of magnitude of the time required to build-up large-scale $M$-energy at a given scale $L$ when $V$-helicity is injected at a rate $\tilde{\epsilon}^V$ per unit mass and at a scale $l_{\text{inj}}$. For this, we take for the inertial range of $M$-helicity the form (5.4) with the order-one numerical constant $C_2$ dropped. Let this range extend from $K = L^{-1}$ to $k_{\text{inj}} = l_{\text{inj}}^{-1}$, then the total $M$-helicity present will be of the order of

$$H^M(L, l_{\text{inj}}) = \int_{1/L}^{1/l_{\text{inj}}} |\tilde{\epsilon}_{\text{tr}}^M|^{\frac{2}{3}} k^{-2} dk \approx (\tilde{\epsilon}_{\text{tr}}^M)^{\frac{2}{3}} L. \tag{6.4}$$

The time required to inject this $M$-helicity is

$$t(L, l_{\text{inj}}) \approx H^M/|\tilde{\epsilon}_{\text{tr}}^M| \approx L|\tilde{\epsilon}_{\text{tr}}^M|^{-\frac{1}{3}}.$$

In view of (6.1) and (6.2) we obtain

$$t(L, l_{\text{inj}}) \approx L(|\tilde{\epsilon}^V| l_{\text{inj}}^2)^{-\frac{1}{3}}. \tag{6.5}$$

It is seen that this time is proportional to the large scale. Equation (6.5) has been found to be in good agreement with our numerical results. Notice that the kinetic helicity effect of the linear dynamo theory already gives a build-up time proportional to the large scale when the kinetic torsality can be considered as prescribed. The point is, however, that in the nonlinear case the growth of large-scale $M$-fields does not result directly from the destabilization by the kinetic helicity injected at small wavenumbers, but is obtained by a cascade which poses no saturation problem.

If this result is to be applied to a realistic problem, homogeneity and isotropy must be given up but (6.5) is probably still valid as an order of magnitude. The simplest mechanism which will generate $V$-helicity is the combination of an overall rotation $\Omega$ and a gradient of turbulent $V$-energy $\langle v^2 \rangle$ with typical scale $l_{\text{grad}}$ (Léorat 1975). The local $V$-helicity injection rate is then found to be of the order of

$$|\tilde{\epsilon}^V| \approx \Omega \langle v^2 \rangle l_{\text{grad}}^{-1}. \tag{6.6}$$

Using (6.6) in (6.5) we obtain for the build-up time $t_B$

$$t_B \approx L\{\Omega \langle v^2 \rangle l_{\text{grad}}^{-1} l_{\text{inj}}^2\}^{-\frac{1}{3}}. \tag{6.7}$$

It must be stressed that dissipation-range quantities such as viscosity and magnetic diffusivity do not appear in this relation.

As an illustration, we consider the time necessary to regenerate the global solar magnetic field starting from zero. We take $l_{\text{grad}}$ and $l_{\text{inj}}$ equal to a typical height scale (a few hundred kilometres); we take velocities of the order of 1 km/s; the large scale $L$ is the radius of the sun. The build-up time is found from (6.7) to be of the order of one year.

## 7. Summary and discussion

This work confirms the importance of helicity both in its kinetic and in its magnetic form for the generation of large-scale magnetic fields by turbulence. We recall that in I investigation of the absolute equilibrium of MHD turbulence suggested the possibility of an inverse cascade of $M$-helicity to small wavenumbers, analogous in certain respects to the inverse cascade of energy in two-dimensional non-magnetic turbulence (Kraichnan 1967; Pouquet *et al.* 1975).

The homogeneous isotropic helical MHD turbulence problem has been investigated using a modification of the eddy-damped quasi-normal Markovian

(EDQNM) approximation (Orszag 1970, 1976). In the MHD case, the eddy-damping rate includes a contribution from Alfvén waves. A set of four coupled integro-differential equations has been obtained in § 2 for the kinetic and magnetic energy and helicity spectra. The cross-helicity spectrum (corresponding to the invariant $H_C = \frac{1}{2}\int \mathbf{v}.\mathbf{b}\, d^3r$) has been taken identically zero, because if it is zero initially it remains so and we are mainly interested in the generation of magnetic fields starting from an infinitesimal seed field.

In § 3, non-local interactions involving triads of wavenumbers $(k, p, q)$ with $k \ll p \sim q$ or $q \ll k \sim p$ were found to be very important in magnetic turbulence; outstanding effects are as follows.

(i) *The Alfvén effect.* Relaxation to zero of the residual energy $E_k^R = E_k^V - E_k^M$ and residual helicity $H_k^R = H_k^V - k^2 H_k^M$ in a time of the order of the period of Alfvén waves produced by large-scale magnetic fields,

(ii) *The kinetic and magnetic helicity effects.* It is found that small-scale helicity destabilizes large-scale magnetic energy and helicity. However, it is not the kinetic helicity alone which acts, as in the Steenbeck *et al.* (1966) theory, but the *residual* helicity. The rate of growth involves the 'torsality' $\alpha_k^R = \alpha_k^V - \alpha_k^M$. The expression for the kinetic torsality $\alpha_k^V$ reduces to that given by Moffatt (1970*a*) when most of the helicity is confined to the dissipation range.

The question of possible energy and helicity cascades and their direction has been considered both theoretically and numerically in situations with large Reynolds number and unit magnetic Prandtl number. In the magnetic non-helical case Kraichnan's (1965) phenomenological theory predicts a $-\frac{3}{2}$ inertial range where $V$- and $M$-energy are in equipartition at each wavenumber and cascade to large wavenumbers. We have indeed obtained in § 4 a $-\frac{3}{2}$ equipartition inertial range but there are $-2$ corrections (and possibly higher ones) leading to an excess of $M$-energy. The same excess of $M$-energy is also found in the direct numerical simulations at much lower Reynolds numbers of Pouquet & Patterson (1976).

In the magnetic helical case, we have first considered in § 5 the case when $M$-helicity is injected into the energy range. An inverse cascade of $M$-helicity is obtained which carries $M$-helicity, $M$-energy and appreciable amounts of $V$-helicity and $V$-energy to ever-larger scales. The total $M$-helicity grows linearly with time; the mean $M$-energy has a slower growth and may possibly saturate. In this inverse cascade, $M$-energy and helicity spectra follow approximately $-1$ and $-2$ power laws. The inverse cascade proceeds as follows: the residual helicity in the energy range, say $k \sim k_E$, coming from the $M$-helicity injection produces a growth by the helicity effect of both $M$-energy and $M$-helicity in smaller wavenumbers, say $k \sim \frac{1}{2}k_E$. The increasing $M$-energy near $\frac{1}{2}k_E$ tends by the Alfvén effect to reduce the residual helicity near $k_E$ whereas the $M$-helicity near $\frac{1}{2}k_E$ destabilizes smaller wavenumbers, say $k \sim \frac{1}{4}k_E$ and so on.

The existence for the full nonlinear MHD equations of an inverse cascade of $M$-helicity with a clearly displayed quasi-stationary inertial range (see figure 4) is a much stronger property than the already recognized existence of inverse transfer of energy by the (kinetic) helicity effect, which is a consequence only of the linear Ohm's law (Roberts 1971).

The cascade mechanism, a competition between the helicity and Alfvén effects leading to an indefinite cascade, is quite different from that envisaged by several authors who were looking for a nonlinear saturation mechanism which would halt the growth of the $M$-field. Moffatt (1970*a*, *b*, 1972) considers a system with inertial helicity waves where dissipation plays an essential role in order not to obtain zero residual helicity (what Moffatt calls cancellation of the helicity effect by non-dissipative Alfvén waves); the largest growth rate is obtained for a preferred wavenumber, in contrast to our case, where the growth of $M$-excitation proceeds step by step until either injection stops or the largest scale available in the medium is attained. As for the saturation mechanism considered by Vainshtein (1972) and Vainshtein & Zeldovich (1972), we refer the reader to remark (5.1).

In § 6 we have obtained a truly nonlinear turbulent dynamo which needs only $V$-energy and helicity injection plus a seed of $M$-field. Again an indefinite cascade of $M$-helicity takes place and the time required to generate $M$-fields of a given scale $L$ is found to be proportional to $L$.

Finally, it remains to ask how relevant the present EDQNM homogeneous isotropic theory may be to (i) the original MHD equations (1.1) and (1.2) and (ii) real flows encountered in nature, particularly in astrophysical situations.

For the first question, there is strong evidence from direct numerical simulation of the MHD equations (Pouquet & Patterson 1976) that our results on inverse transfer of $M$-helicity and energy are not spurious effects introduced by the EDQNM approximation. In this direct numerical simulation of decaying helical MHD turbulence, which uses a modified Orszag–Patterson (1972) scheme with $32^3$ Fourier modes, the transfer of $M$-energy to small wavenumbers is found to be persistently positive when $M$-helicity is present, whereas it is negative in the non-helical case.

The second question raises problems such as: what happens if one includes anisotropy, inhomogeneity, rotation, boundary conditions, compressibility, etc.? Compressibility generally destroys the $V$-helicity invariant in the non-magnetic case but it does not destroy the $M$-helicity invariant, which requires only Ohm's law. Since the basic ingredient of the inverse cascade is the conservation of the $M$-helicity, the build-up of large-scale fields by an inverse cascade mechanism is not ruled out in compressible media. Anisotropies, inhomogeneities, rotation and boundary conditions can, in principle, be dealt with by the EDQNM method, although the exact expression for the eddy-damping rate is somewhat uncertain and a more complete theory like the generalized TFM (Kraichnan 1972) would in principle be preferable. However, the algebraic and numerical work implied is so formidable that this must be ruled out. Anyhow, it is likely that in weakly anisotropic and inhomogeneous situations the overall feature of MHD turbulence will not be upset, in particular the $-\frac{3}{2}$ non-helical inertial range and the expression (6.5) for the build-up time of large-scale fields. To cope with rotation, boundary conditions and non-prescribed systematic velocity fields, the simplest procedure may be a parametrization of the small-scale turbulence like that done, for example, by Malkus & Proctor (1975). However, certain precautions are required: turbulent eddy diffusivities must be used and it is not legitimate to assume a given torsality $\alpha$ since Alfvén waves propagating in the large-scale field $\mathbf{B}_0$ (random or deterministic) will relax the residual torsality to zero in a time $\sim|B_0|^{-1}$.

We are grateful to J. C. André, J. Herring and R. Kraichnan for helpful advice. We also thank for its generous hospitality the National Centre for Atmospheric Research, where most of the numerical calculations were done on a CDC 6600/7600.

## Appendix

*Geometric coefficients*

Let $\alpha$, $\beta$ and $\gamma$ be the angles opposite to the sides $k$, $p$ and $q$ of a triangle, and let $x$, $y$ and $z$ be their cosines. The geometric coefficients appearing in the spectral equations are

$$\left.\begin{aligned} b_{kpq} &= (p/k)(xy+z^3), & c_{kpq} &= (p/k)z(1-y^2), \\ h_{kpq} &= (p/k)(z+xy) = 1-y^2, & j_{kpq} &= (p/k)z(1-x^2), \end{aligned}\right\} \tag{A 1}$$

$$e_{kpq} = x(1-z^2), \qquad f_{kpq} = z-xy-2zy^2. \tag{A 2}$$

Coefficients (A 1) are identical to those of Kraichnan & Nagarajan (1967). They can all be written solely in terms of $b_{kpq}$ and $c_{kpq}$:

$$\left.\begin{aligned} h_{kpq} &= c_{kpq}+(p^2/q^2)\,c_{kqp}, & j_{kpq} &= (p^2/k^2)\,c_{pkq}, \\ e_{kpq} &= (p/q)\,c_{pqk}, & f_{kpq} &= (k/p)\,c_{kpq}-(kp/q^2)\,c_{kqp}. \end{aligned}\right\} \tag{A 3}$$

Other useful relations are (see also Kraichnan & Nagarajan 1967)

$$k^2b_{kpq} = p^2b_{pkq}, \quad k^2h_{kpq} = p^2h_{pkq}, \tag{A 4}$$

$$qf_{kpq}+pf_{kqp} = 0, \qquad k^4c_{kpq} = p^4c_{pkq}. \tag{A 5}$$

*Expansions for non-local interactions*

Calculations are more easily carried out in polar co-ordinates with integration variables $(q, \beta)$ or $(p, \gamma)$ instead of the bipolar co-ordinates $(p, q)$. The volume element must be changed according to

$$dp\,dq = (kq/p)\sin\beta\,d\beta\,dq = (kp/q)\sin\gamma\,d\gamma\,dp. \tag{A 6}$$

In obtaining the non-local contributions to the transfer, frequent use is made of the following formulae, for which three cases must be distinguished.

(*a*) $k \ll p \simeq q$

$$p = q\{1-kq^{-1}\cos\beta+O(k^2q^{-2})\}, \tag{A 7}$$

$$x = 1-\tfrac{1}{2}k^2q^{-2}\sin^2\beta+O(k^3q^{-3}), \quad y = \cos\beta, \tag{A 8}$$

$$z = -\cos\beta+kq^{-1}\sin^2\beta+O(k^2q^{-2}), \tag{A 9}$$

$$dp\,dq = k\{\sin\beta+kq^{-1}\sin\beta\cos\beta+O(k^2q^{-2})\}\,dq\,d\beta, \tag{A 10}$$

$$(xy+z^3)\,dp\,dq = k\{\cos\beta\sin^3\beta+4kq^{-1}\sin^3\beta\cos^2\beta+O(k^2q^{-2})\}\,dq\,d\beta, \tag{A 11}$$

$$z(1-y^2)\,dp\,dq = k\{-\cos\beta\sin^3\beta + kq^{-1}\sin^5\beta - kq^{-1}\sin^3\beta\cos^2\beta + O(k^2q^{-2})\}\,dq\,d\beta, \quad \text{(A 12)}$$

$$(z+xy)\,dp\,dq = k\{kq^{-1}\sin^3\beta + O(k^2q^{-2})\}\,dq\,d\beta, \quad \text{(A 13)}$$

$$z(1-x^2)\,dp\,dq = k\{-k^2q^{-2}\sin^3\beta\cos\beta + O(k^3q^{-3})\}\,dq\,d\beta, \quad \text{(A 14)}$$

$$x(1-z^2)\,dp\,dq = k\{\sin^3\beta + 3kq^{-1}\sin^3\beta\cos\beta + O(k^2q^{-2})\}\,dq\,d\beta, \quad \text{(A 15)}$$

$$(z-xy-2zy^2)\,dp\,dq = k\{-2\cos\beta\sin^3\beta + kq^{-1}\sin^3\beta(1-4\cos^2\beta) + O(k^2q^{-2})\}\,dq\,d\beta. \quad \text{(A 16)}$$

(b) $p \ll k \simeq q$

$$q = k\{1 - pk^{-1}\cos\gamma + O(p^2k^{-2})\}, \quad \text{(A 17)}$$

$$x = -\cos\gamma + pk^{-1}\sin^2\gamma + O(p^2k^{-2}), \quad \text{(A 18)}$$

$$y = 1 - \tfrac{1}{2}p^2k^{-2}\sin^2\gamma + O(p^3k^{-3}), \quad z = \cos\gamma, \quad \text{(A 19)}$$

$$dp\,dq = p\{\sin\gamma + pk^{-1}\sin\gamma\cos\gamma + O(p^2k^{-2})\}\,dp\,d\gamma, \quad \text{(A 20)}$$

$$(xy+z^3)\,dp\,dq = p\{-\cos\gamma\sin^3\gamma + pk^{-1}\sin^5\gamma + O(p^2k^{-2})\}\,dp\,d\gamma, \quad \text{(A 21)}$$

$$z(1-y^2)\,dp\,dq = p\{p^2k^{-2}\sin^3\gamma\cos\gamma + O(p^3k^{-3})\}\,dp\,d\gamma, \quad \text{(A 22)}$$

$$(z+xy)\,dp\,dq = p\{pk^{-1}\sin^3\gamma + O(p^2k^{-2})\}\,dp\,d\gamma, \quad \text{(A 23)}$$

$$z(1-x^2)\,dp\,dq = p\{\cos\gamma\sin^3\gamma + 3pk^{-1}\sin^3\gamma\cos^2\gamma + O(p^2k^{-2})\}\,dp\,d\gamma, \quad \text{(A 24)}$$

$$x(1-z^2)\,dp\,dq = p\{-\cos\gamma\sin^3\gamma + pk^{-1}(\sin^4\gamma - \sin^3\gamma\cos^2\gamma) + O(p^2k^{-2})\}\,dp\,d\gamma, \quad \text{(A 25)}$$

$$(z-xy-2zy^2)\,dp\,dq = p\{-pk^{-1}\sin^3\gamma + O(p^2k^{-2})\}\,dp\,d\gamma. \quad \text{(A 26)}$$

(c) $q \ll k \simeq p$

$$p = k\{1 - qk^{-1}\cos\beta + O(q^2k^{-2})\}, \quad \text{(A 27)}$$

$$x = -\cos\beta + qk^{-1}\sin^2\beta + O(q^2k^{-2}), \quad y = \cos\beta, \quad \text{(A 28)}$$

$$z = 1 - \tfrac{1}{2}q^2k^{-2}\sin^2\beta + O(q^2k^{-2}), \quad \text{(A 29)}$$

$$dp\,dq = q\{\sin\beta + qk^{-1}\sin\beta\cos\beta + O(q^2k^{-2})\}\,dq\,d\beta, \quad \text{(A 30)}$$

$$(xy+z^3)\,dp\,dq = q\{\sin^3\beta + 2qk^{-1}\sin^3\beta\cos\beta + O(q^2k^{-2})\}\,dq\,d\beta, \quad \text{(A 31)}$$

$$z(1-y^2)\,dp\,dq = q\{\sin^3\beta + qk^{-1}\sin^3\cos\beta + O(q^2k^{-2})\}\,dq\,d\beta, \quad \text{(A 32)}$$

$$(z+xy)\,dp\,dq = q\{\sin^3\beta + 2qk^{-1}\sin^3\beta\cos\beta + O(q^2k^{-2})\}\,dq\,d\beta, \quad \text{(A 33)}$$

$$z(1-x^2)\,dp\,dq = q\{\sin^3\beta + 3qk^{-1}\sin^3\beta\cos\beta + O(q^2k^{-2})\}\,dq\,d\beta, \quad \text{(A 34)}$$

$$x(1-z^2)\,dp\,dq = q\{-q^2k^{-2}\sin^3\beta\cos\beta + O(q^3k^{-3})\}\,dq\,d\beta, \quad \text{(A 35)}$$

$$(z-xy-2zy^2)\,dp\,dq = q\{\sin^3\beta + O(q^2k^{-2})\}\,dq\,d\beta. \quad \text{(A 36)}$$

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† This and subsequent papers appear in English translation in Roberts & Stix (1971).